

Emergent Schwarzschild : symplectic, geometric and topological perspective

Sumanto Chanda ♣ , Partha Guha ♣ and Raju Roychowdhury ♠

♣ *S.N. Bose National Centre for Basic Sciences
JD Block, Sector III, Salt Lake, Kolkata 700098, India*

♠ *Instituto de Fisica, Universidade de Sao Paulo,
C. Postal 66318, 05314-970 Sao Paulo, SP, Brazil*

E-mail: sumanto12@boson.bose.res.in, partha@boson.bose.res.in,
raju.roychowdhury@gmail.com

ABSTRACT: In the bottom-up approach of emergent gravity we attempt to find out symplectic gauge fields emerging from Euclidean Schwarzschild metric which is a Ricci flat manifold but non-Kähler. The Euclidean Schwarzschild solution is studied extensively as a electromagnetism defined on the symplectic space (M, ω) . A detailed geometrical engineering with the emergent metric help us set up the Seiberg Witten map between commutative and non- commutative gauge fields and further paves the path for the evaluation of the topological invariants in terms of gauge theory quantities.

KEYWORDS: Emergent Gravity, Euclidean Schwarzschild, Darboux theorem, Symplectomorphism, Bianchi identity, Topological Invariants.

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1. Introduction

It is worth paying attention to the fact that the thermal nature of black hole emission can be related directly to the properties of Euclidean Schwarzschild solution ala Hawking. In the Euclidean approach to quantum field theory one attempts to define quantities on a “Euclidean section” and then obtain the physical spacetime quantities by analytic continuation. Particularly, the Feynman propagator for a field on spacetime is obtained by analytically continuing the Green’s function on Euclidean section. Thus one is naturally led to study and examine the salient features of Euclidean Schwarzschild solution.

Mathematically the Euclidean Schwarzschild 4-manifold M is a complete solution to the Euclidean Einstein’s equations with zero cosmological constant, and has the non-trivial topology $M \cong \mathbb{R}^2 \times S^2$. In other words it is a Ricci flat manifold. It is not a gravitational instanton (such as e.g. the Taub-NUT metric or the Eguchi-Hanson metric) in that its curvature tensor is not self-dual.

We have a particularly nice form of the metric g on a dense open subset $(\mathbb{R}^2 \setminus \{O\}) \times S^2 \subset M \cong \mathbb{R}^2 \times S^2$ of the Euclidean Schwarzschild manifold. It is convenient to use polar coordinates (r, τ) on $\mathbb{R}^2 \setminus \{O\}$ in the range $r \in (2m, \infty)$ and $\tau \in [0, 8\pi m)$, where $m > 0$ is a fixed constant related to the mass of the black hole. The metric then takes the form

$$ds^2 = \left(1 - \frac{2m}{r}\right) d\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

where $d\Omega^2$ stands for the line element of the unit round S^2 . In spherical coordinates $\Theta \in (0, \pi)$ and $\phi \in [0, 2\pi)$ it is

$$d\Omega^2 = d\Theta^2 + \sin^2 \Theta d\phi^2$$

on the open coordinate chart $(S^2 \setminus (\{S\} \cup \{N\})) \subset S^2$. Consequently the above metric takes the following form on the open, dense coordinate chart $U := (\mathbb{R}^2 \setminus \{O\}) \times (S^2 \setminus (\{S\} \cup \{N\})) \subset M \cong \mathbb{R}^2 \times S^2$:

$$ds^2 = \left(1 - \frac{2m}{r}\right) d\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\Theta^2 + \sin^2 \Theta d\phi^2). \quad (1.1)$$

Despite the apparent singularity of the metric at the origin $O \in \mathbb{R}^2$, it can be extended analytically to the whole $\mathbb{R}^2 \times S^2$ as demonstrated in Wald [1].

The $U(1)$ action defined by $\tau \mapsto \tau + 4m\lambda$ for $e^{i\lambda} \in U(1)$ leaves this metric invariant, and thus defines the Killing vector field

$$X := \frac{1}{4m} \frac{\partial}{\partial \tau},$$

which (together with the $U(1)$ action itself) clearly extends to a Killing field on the whole Euclidean Schwarzschild manifold, which we will denote by X .

Now consider the differential 1-form $\xi := g(X, \cdot)$ dual to X . In our coordinate chart U it takes the form

$$\xi = \frac{1}{4m} \left(1 - \frac{2m}{r}\right) d\tau.$$

General considerations about Killing's equations on a Ricci flat manifold yield that $d\xi$ is a harmonic 2-form, which on a complete manifold is equivalent to saying that it is closed and coclosed and thus harmonic.

The correspondence between noncommutative (NC) $U(1)$ gauge theory and gravity has gained much attention in the context of emergent gravity [2, 3, 4, 5, 6]. Current research in the field of instantons [7, 8] reveals that the gravitational instantons in Einstein gravity are equivalent to $U(1)$ instantons in NC gauge theory. In other words, the self-dual electromagnetism on NC spacetime is equivalent to self-dual Einstein gravity [9]. This implies that gravity can emerge from electromagnetism defined in NC spacetime. The relation between Yang-Mills instantons and gravitational instantons are further understood in [10] where it was shown that every gravitational instantons are $SU(2)$ Yang-Mills instantons on a Ricci-flat four manifold but the reverse is not necessarily true. Gravitational instantons satisfy the same self-dual equations of $SU(2)$ Yang-Mills instantons. The gravitational

instanton which is a solution of (anti) self-dual gravity emerges either from $SU(2)_L$ or $SU(2)_R$ Yang-Mills instanton sector. The (anti) self-dual gauge fields constructed from Yang-Mills instanton generate (anti) self-dual gravity. In [10] the result was further extended to include general Einstein manifolds [11]: all Einstein manifolds with or without cosmological constant are Yang-Mills instantons in $O(4) = SU(2)_L \times SU(2)_R$ gauge theory but the reverse is not true. In fact they arise as a sum of instantons coming both from $SU(2)_L$ instanton and $SU(2)_R$ anti-instanton. This may explain the stability of the four dimensional Einstein manifold compared to the five dimensional Kaluza-Klein vacuum.

In this note we deal with a specific example of an Einstein manifold: the Euclidean Schwarzschild black hole. It is an Einstein manifold which is Ricci flat. It is argued that this geometry is a sum of both $SU(2)_L$ and $SU(2)_R$ instanton. However, the euclidean Schwarzschild solution is not gravitational instanton in the sense that this black hole is not a solution of the (anti) self-dual gravity. It was discussed in [10] that the Euclidean solution outside of the (anti) self-dual gravity is a combination of both $SU(2)_L$ and $SU(2)_R$ Yang-Mills instanton. Following the bottom-up approach of Emergent Gravity [12], we construct vector fields from the Euclidean Schwarzschild black hole and calculate the equations of motion and Jacobi identity. Using the Seiberg Witten map we find the symplectic field strength and check the absence of self-duality for Euclidean Schwarzschild. We explicitly show the Ricci flatness and shed light on the vacuum Einstein solution as is evident from the energy momentum tensor that can be computed exactly exploiting the relation between spin connections and structure constants for the Schwarzschild solution. We further study their geometric properties by calculating the topological invariants of the $U(1)$ gauge fields [13] derived from emergent Schwarzschild metric.

The paper is organized as follows: In section 2. we review the standard results of the bottom-up formulation of emergent gravity [12], In section 3. we introduce the euclidean Schwarzschild metric, we make a wise choice for the Darboux coordinates in which we write the corresponding metric, then we obtain the set of symplectic $U(1)$ gauge fields and derive the corresponding vector fields and check the Jacobi identity for the Poisson and Lie algebra. Next we realize the Seiberg Witten map between ordinary and NC gauge fields and find that the solution is neither self-dual nor anti self-dual and hence not a gravitational instanton. In the next section, from the set of vierbeins we obtain the spin-connections and the curvature components and from that we get the Ricci tensor and obtain Ricci flatness for the metric. Ricci flatness condition also translates into a vacuum solution. In the penultimate section, we compute the bulk and boundary contribution to the topological invariants namely Euler characteristics and the Hirzebruch signature complex. Here we also obtain $SU(2)_\pm$ gauge fields for emergent Schwarzschild metric and reconfirm the fact that both the gauge fields make an equal contribution to the overall Euler invariant or the signature. Thus emergent Schwarzschild can be seen as the sum of $SU(2)_L$ instantons and $SU(2)_R$ anti-instantons, thus explaining the generic feature of stability for a Ricci-flat manifold like the one we dealt with. We conclude with some comments and future directions. The appendices contain some details of the computations, namely some identities from differential geometry that has been used, also the t'Hooft matrices and the full set of $SU(2)_\pm$ gauge fields in matrix notation.

2. Review of Standard Results

The mathematical tool to quantize the dynamical system [14] is to specify the Poisson structure θ such that

$$\theta = \frac{1}{2} \sum_{A,B=1}^N \theta^{AB} \frac{\partial}{\partial x^A} \wedge \frac{\partial}{\partial x^B} \in \Gamma(\wedge^2 TM), \quad (2.1)$$

and then the differentiable manifold M endowed with θ describes a Poisson manifold (M, θ) . The Poisson structure defines an **R**-bilinear antisymmetric operation $\{, \}_\theta: C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ by

$$(f, g) \mapsto \{f, g\}_\theta = \langle \theta, df \otimes dg \rangle = \theta^{AB}(x) \frac{\partial f(x)}{\partial x^A} \frac{\partial g(x)}{\partial x^B}, \quad (2.2)$$

and the Poisson bracket satisfy the Leibniz rule and Jacobi identity as follows:

$$\{f, gh\}_\theta = g\{f, h\}_\theta + \{f, g\}_\theta h, \quad (2.3)$$

$$\{f, \{g, h\}_\theta\}_\theta + \{g, \{h, f\}_\theta\}_\theta + \{h, \{f, g\}_\theta\}_\theta = 0, \quad (2.4)$$

$\forall f, g, h \in C^\infty(M)$. The Poisson structure θ reduces to symplectic structure when it is nondegenerate.

The application of Darboux theorem or Moser lemma[14] of symplectic geometry to electromagnetism defined on the symplectic space gives rise to an equivalence principle. An arbitrary deformation of symplectic deformation can not be distinguishable locally from canonical form. The electromagnetism on symplectic spacetime can be a theory of gravity[15]: Starting with symplectic form $\omega_0 = B$, the deformation of ω_0 generate dynamical gauge fields such that $\omega_1 = B + F$, where $F = dA$. It is always possible to eliminate F by a suitable coordinate transformation as far as the 2-form B is closed and nondegenerate because in this case the gauge symmetry becomes a spacetime symmetry rather than an internal symmetry. This very fact indeed paves the way for a connection between NC gauge fields and spacetime geometry.

For a given Poisson algebra $(C^\infty(M), \{, \}_\theta)$, there is a natural map $C^\infty(M) \rightarrow TM$: $f \mapsto X_f$ between smooth functions in $C^\infty(M)$ and vector fields in TM such that

$$X_f(g)(y) \equiv \{g, f\}_\theta(y) = \left(\theta^{\mu\nu} \frac{\partial f(y)}{\partial y^\nu} \frac{\partial}{\partial y^\mu} \right) g(y), \quad (2.5)$$

for any $g \in C^\infty(M)$. This means that we can obtain a vector field $X_f = X_f^\mu \partial_\mu \in \Gamma(TM_y)$ from a smooth function $f \in C^\infty(M)$ defined at $y \in M$ where $X_f^\mu(y) = \theta^{\mu\nu} \frac{\partial f(y)}{\partial y^\nu}$. As long as θ is a Poisson structure of M , the above formula (2.5) between Hamiltonian function f and Hamiltonian vector field X_f is a Lie algebra homomorphism in the sense that

$$X_{\{f, g\}_\theta} = -[X_f, X_g], \quad (2.6)$$

where the right hand side is a Lie bracket between Hamiltonian vector fields.

From the above arguments, $U(1)$ gauge fields on a symplectic manifold $(M, B = \theta^{-1})$ can be transformed into a set of smooth functions

$$\{D_\mu(y) \in C^\infty(M) | D_\mu(y) \equiv B_{\mu\nu}x^\nu(y) = B_{\mu\nu}y^\nu + \hat{A}_\mu(y), \mu, \nu = 1, \dots, 2n\}, \quad (2.7)$$

where $x^\mu(y) \equiv y^\mu + \theta^{\mu\nu} \hat{A}_\nu(y) \in C^\infty(M)$.

After the map (2.5) is applied, we obtain Lie algebra homomorphism (2.6) between the Poisson algebra $(C^\infty(M), \{\cdot, \cdot\}_\theta)$ and the Lie algebra $(\Gamma(TM), [\cdot, \cdot])$ of vector fields defined by

$$\{V_\mu = V_\mu^a \partial_a \in \Gamma(TM) | V_\mu(f)(y) \equiv \{D_\mu(y), f(y)\}_\theta, a = 1, \dots, 2n\}, \quad (2.8)$$

for any $f \in C^\infty(M)$. The vector fields $V_\mu = V_\mu^a(y) \frac{\partial}{\partial y^a} \in \Gamma(TM_y)$ take values in the Lie algebra of volume preserving diffeomorphisms ($\partial_a V_\mu^a = 0$). However, it can be shown that the vector fields $V_\mu \in \Gamma(TM)$ are related to the orthonormal frames (vielbeins) E_μ by $V_\mu = \lambda E_\mu$ where $\lambda^2 = \det V_\mu^a$. The metric is constructed from this vector fields:

$$ds^2 = \delta_{\mu\nu} E^\mu \otimes E^\nu = \lambda^2 \delta_{\mu\nu} V_\mu^a V_\nu^b dy^a \otimes dy^b, \quad (2.9)$$

where $E^\mu = \lambda V^\mu \in \Gamma(T^*M)$ are dual one-forms.

The electromagnetic fields in the symplectic spacetime (M, B) manifest themselves only as a deformation of symplectic structure such that the resulting symplectic spacetime is described by $(M, B + F)$ where $F = dA = L_X B$. This is equivalent to a deformation of frame bundle over spacetime manifold M : $\partial_\mu \rightarrow E_\mu = E_\mu^a(y) \partial_a$, or, in terms of dual frames, $dy^\mu \rightarrow E^\mu = E_a^\mu(y) dy^a$. That is

$$ds^2 = \delta_{\mu\nu} dy^\mu \otimes dy^\nu \rightarrow ds^2 = \delta_{\mu\nu} E^\mu \otimes E^\nu. \quad (2.10)$$

We can show the emergence of gravity from the gauge fields starting with the action:

$$S_p = \frac{1}{4g_{YM}^2} \int d^{2n}y \{D_\mu(y), D_\nu(y)\}_\theta \{D^\mu(y), D^\nu(y)\}_\theta. \quad (2.11)$$

where g_{YM} is a $2n$ -dimensional gauge coupling constant. Note that

$$\{D_\mu(y), D_\nu(y)\}_\theta = -B_{\mu\nu} + \partial_\mu \hat{A}_\nu(y) - \partial_\nu \hat{A}_\mu(y) + \{\hat{A}_\mu(y), \hat{A}_\nu(y)\}_\theta, \quad (2.12)$$

$$\equiv -B_{\mu\nu} + \hat{F}_{\mu\nu}(y), \quad (2.13)$$

and

$$\{D_\mu(y), \{D_\nu(y), D_\lambda(y)\}_\theta\}_\theta = \partial_\mu \hat{F}_{\nu\lambda}(y) + \{\hat{A}_\mu(y), \hat{F}_{\nu\lambda}(y)\}_\theta, \quad (2.14)$$

$$\equiv \hat{D}_\mu \hat{F}_{\nu\lambda}(y). \quad (2.15)$$

By identifying $f(y) = D_\mu(y)$ and $g(y) = D_\nu(y)$ with the relation of (2.13), the Lie algebra homomorphism (2.6) leads to the following identity

$$X_{\hat{F}_{\mu\nu}} = [V_\mu, V_\nu], \quad (2.16)$$

where $V_\mu \equiv X_{D_\mu}$ and $V_\nu \equiv X_{D_\nu}$ and using (2.15) we have

$$X_{\hat{D}_\mu \hat{F}_{\nu\lambda}} = [V_\mu, [V_\nu, V_\lambda]]. \quad (2.17)$$

Thus the equation of motion and the Jacobi identity can be written as

$$\{D^\mu, \{D_\mu, D_\nu\}_\theta\}_\theta = \hat{D}^\mu \hat{F}_{\mu\nu} = 0, \quad (2.18)$$

$$\{D_{[\mu}, \{D_\nu, D_\lambda]\}_\theta\}_\theta = \hat{D}_{[\mu} \hat{F}_{\nu\lambda]} = 0. \quad (2.19)$$

With the help of the above formula we have the following insightful correspondence

$$\hat{D}_{[\mu} \hat{F}_{\nu\lambda]} = 0 \Leftrightarrow [V_{[\mu}, [V_\nu, V_\lambda]] = 0, \quad (2.20)$$

$$\hat{D}^\mu \hat{F}_{\mu\nu} = 0 \Leftrightarrow [V^\mu, [V_\mu, V_\nu]] = 0. \quad (2.21)$$

These relations reduce to the Einstein field equations and the first Bianchi identity for the Riemann tensor

$$[V^\mu, [V_\mu, V_\nu]] = 0 \Leftrightarrow R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad (2.22)$$

$$[V_{[\mu}, [V_\nu, V_\lambda]] = 0 \Leftrightarrow R_{[\mu\nu\lambda]\rho} = 0. \quad (2.23)$$

where the 2nd equation above hints that the individual Riemann curvature components can be given by

$$[V_\mu, [V_\nu, V_\lambda]] = R_{\mu\nu\lambda}{}^\rho V_\rho \quad (2.24)$$

This equation will be of relevance to us later, in the next section as we shall see.

3. Euclidean Schwarzschild Gauge Fields

The Euclidean Schwarzschild metric is given by:

$$ds^2 = f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (3.1)$$

where $f(r) = 1 - \frac{2m}{r}$

In this section, we will study the symplectic gauge fields corresponding to this metric, and then will study the geometry of the vector field vierbeins arising from the gauge fields, and verify if it is self dual or not. The last step will reveal to us if it is a gravitational instanton. For now, our first requirement will be to construct a new co-ordinate chart that will serve our purpose.

3.1 The Darboux chart

The Darboux Theorem states that we can always locally eliminate dynamical gauge fields that fluctuate about the background vacuum condensate through a local co-ordinate transformation. In general relativity, the Equivalence Principle states that there always exists a diffeomorphism that equates a curved manifold locally to a flat manifold. This theorem applies for Riemannian manifolds.

Thus, the Darboux Theorem is the equivalence principle for Symplectic manifolds. It essentially states that the symplectic structure on a curved manifold can always be equated to the symplectic structure on a flat manifold via a diffeomorphism. It can be summed up by the mathematical statement below:

$$\exists \quad \frac{\partial y^\mu}{\partial \xi^a} \quad s.t. \quad \mathcal{F}_{\mu\nu}(x) \frac{\partial y^\mu}{\partial \xi^a} \frac{\partial y^\nu}{\partial \xi^b} = B_{ab} \quad (3.1.1)$$

The question here is what kind of diffeomorphism will satisfy equation (3.1.1). The crudest answer we can give so far requires that we first write the perturbed symplectic structure as:

$$\mathcal{F}_{\mu\nu}(x) = B_{\mu\nu} + \lambda F_{\mu\nu}(x) \quad (3.1.2)$$

such that the parameter λ sets the strength of the dynamical field perturbation to the symplectic structure.

In the case of a given metric, we can compute the individual curvature components. Embedded within the curvature are the various $SU(2)_\pm$ gauge field components.

$$R_{ab} = \eta_{ab}^{i(+)} F^{i(+)} + \eta_{ab}^{i(-)} F^{i(-)} \quad \Rightarrow \quad F^{i(\pm)} = \frac{1}{4} \eta_{ab}^{i(\pm)} R_{ab} \quad (3.1.3)$$

The simplest way to eliminate local dynamical gauge fields upon switching to the Darboux co-ordinates, is to eliminate the individual $SU(2)_\pm$ gauge fields. This is necessarily true as we shall see below. It is known that in maximally symmetric spaces, we can have the curvature in the form:

$$R_{abcd} = g^{ij}(\vec{x}) \varepsilon_{iab} \varepsilon_{jcd} \quad (3.1.4)$$

In the case of self-dual curvature and fields, we can further elaborate it as:

$$R_{ab} = \alpha_{ij}^{(+)}(\vec{x}) \eta_{ab}^{i(+)} \eta_{cd}^{j(+)} + \alpha_{ij}^{(-)}(\vec{x}) \eta_{ab}^{i(-)} \eta_{cd}^{j(-)} \quad \Rightarrow \quad F^{i(\pm)} = \frac{1}{2} \alpha^{ij(\pm)} \eta_{ab}^{j(\pm)} e^a \wedge e^b \quad (3.1.5)$$

where all the $\alpha^{ij(\pm)}(\vec{x})$ tensor components are diagonal (ie. $\alpha^{ij(\pm)}(\vec{x}) = 0$ for $i \neq j$). This means that the dynamical gauge field strength affiliated with the metric as a linear combination of the individual components using the t'Hooft symbols as a basis.

$$\begin{aligned} F &= c^{i(+)} F^{i(+)} + c^{i(-)} F^{i(-)} \\ \Rightarrow \quad F_{ab} &= c^{i(+)} \alpha^{ij(+)}(\vec{x}) \eta_{ab}^{j(+)} + c^{i(-)} \alpha^{ij(-)}(\vec{x}) \eta_{ab}^{j(-)} \end{aligned} \quad (3.1.6)$$

Now, since these t'Hooft symbols never share the same non-zero matrix elements in the same positions, we can say that the $SU(2)_\pm$ gauge fields are linearly independent 2-forms. From linear algebra, we know that this implies that:

$$F = 0 \quad \longleftrightarrow \quad \alpha^{ij(\pm)}(\vec{x}) \quad \Rightarrow \quad F^{i(\pm)} = 0 \quad \longleftrightarrow \quad R_{ab} = 0 \quad (3.1.7)$$

This consequently eliminates the curvature as well, which describes the equivalence principle. Thus, **if we can choose a local co-ordinate frame that locally eliminates the curvature, we will also have found the Darboux co-ordinates.**

We need local co-ordinates to obtain and analyse the gauge fields related to the metric. To do this, we could define a local co-ordinate system which preserves the volume element formed by the vierbeins of (3.1).

$$\begin{aligned} \nu &= \nu' \\ \Rightarrow \quad e^1 \wedge e^2 \wedge e^3 \wedge e^4 &= \epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4 \\ &= dt \wedge (r^2 dr) \wedge (\sin \theta d\theta) \wedge d\phi \end{aligned} \quad (3.1.8)$$

These co-ordinates are known as the Darboux co-ordinates, the principle behind this design being to make the vierbeins equivalent to the exact differentials of the local choice of co-ordinates.

$$X^a = \{\tau, \rho, x, y\} = \left\{ t, \frac{r^3}{3}, -\cos \theta, \phi \right\} \quad (3.1.9)$$

The metric, in these co-ordinates are then written as:

$$\begin{aligned} ds^2 &= \tilde{f}(\rho) dt^2 + \frac{1}{\tilde{f}(\rho)} dr^2 + r^2(\rho) (d\theta^2 + \sin^2 \theta d\phi^2) \\ &= \tilde{f}(\rho) \left(\frac{\partial t}{\partial \tau} \right)^2 d\tau^2 + \frac{1}{\tilde{f}(\rho)} \left(\frac{\partial r}{\partial \rho} \right)^2 d\rho^2 + r(\rho) \left\{ \left(\frac{\partial \theta}{\partial x} \right)^2 dx^2 + (1 - x^2) \left(\frac{\partial \phi}{\partial y} \right)^2 dy^2 \right\} \\ \therefore \quad ds^2 &= \tilde{f}(\rho) d\tau^2 + \frac{1}{\tilde{f}(\rho)} \frac{d\rho^2}{(3\rho)^{\frac{4}{3}}} + (3\rho)^{\frac{2}{3}} \left\{ \frac{dx^2}{1 - x^2} + (1 - x^2) dy^2 \right\} \end{aligned} \quad (3.1.10)$$

And for the inverse vierbeins we have:

$$\left(\frac{\partial}{\partial s} \right)^2 = \mathcal{E}_a \otimes \mathcal{E}_a = \lambda^{-2} V_a \otimes V_a \quad (3.1.11)$$

$$\begin{aligned} \left(\frac{\partial}{\partial s} \right)^2 &= \tilde{f}^{-1}(\rho) \left(\frac{\partial}{\partial t} \right)^2 + \tilde{f}(\rho) \left(\frac{\partial}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial}{\partial \phi} \right)^2 \\ &= \tilde{f}^{-1}(\rho) \left(\frac{\partial \tau}{\partial t} \right)^2 \left(\frac{\partial}{\partial \tau} \right)^2 + \tilde{f}(\rho) \left(\frac{\partial \rho}{\partial r} \right)^2 \left(\frac{\partial}{\partial \rho} \right)^2 + \frac{1}{r^2} \left(\frac{\partial x}{\partial \theta} \right)^2 \left(\frac{\partial}{\partial x} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial y}{\partial \phi} \right)^2 \left(\frac{\partial}{\partial y} \right)^2 \\ &= \tilde{f}^{-1}(\rho) \left(\frac{\partial}{\partial \tau} \right)^2 + \tilde{f}(\rho) (r^2)^2 \left(\frac{\partial}{\partial \rho} \right)^2 + \frac{\sin^2 \theta}{r^2} \left(\frac{\partial}{\partial x} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial y}{\partial \phi} \right)^2 \left(\frac{\partial}{\partial y} \right)^2 \\ \therefore \left(\frac{\partial}{\partial s} \right)^2 &= \tilde{f}^{-1}(\rho) \left(\frac{\partial}{\partial \tau} \right)^2 + \tilde{f}(\rho) (3\rho)^{\frac{4}{3}} \left(\frac{\partial}{\partial \rho} \right)^2 \\ &\quad + \frac{1}{(3\rho)^{\frac{2}{3}}} \left\{ (1 - x^2) \left(\frac{\partial}{\partial x} \right)^2 + \frac{1}{(1 - x^2)} \left(\frac{\partial}{\partial y} \right)^2 \right\} \end{aligned} \quad (3.1.12)$$

$$\text{where} \quad \tilde{f}(\rho) = 1 - \frac{2m}{(3\rho)^{\frac{1}{3}}}$$

Looking at the metric (3.1) again, one can easily write the two matrices:

$$\epsilon^a = \begin{pmatrix} f^{\frac{1}{2}}(r) & 0 & 0 & 0 \\ 0 & f^{-\frac{1}{2}}(r) & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \sin \theta \end{pmatrix} \quad \mathcal{E}_a = \begin{pmatrix} f^{-\frac{1}{2}}(r) & 0 & 0 & 0 \\ 0 & f^{\frac{1}{2}}(r) & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r \sin \theta} \end{pmatrix} \quad (3.1.13)$$

Using the Darboux co-ordinates of (3.1.9), we can define a symplectic form:

$$\omega = \epsilon^1 \wedge \epsilon^2 + \epsilon^3 \wedge \epsilon^4 = d\tau \wedge d\rho + dx \wedge dy \quad (3.1.14)$$

$$= r^2 dt \wedge dr + \sin \theta \, d\theta \wedge d\phi \quad (3.1.15)$$

such that one can re-obtain the original volume form ν of (3.1.8) as:

$$\nu = \frac{1}{2} \omega \wedge \omega = r^2 \sin \theta \quad (3.1.16)$$

Complex Stereographic Projection - an alternate choice of coordinates

Now there is an understanding that the polar co-ordinate system chosen here results in a multi-valuedness towards the poles that causes a breakdown of the one-to-one correspondence between the cartesian and polar variables, certifying a diffeomorphism, since the azimuthal angle ϕ is now arbitrary.

$$(x, y, z) \longleftrightarrow (r, \theta, \phi) \quad (0, 0, \pm r) \longleftrightarrow (r, 0, ?)$$

Thus, one needs to consider an alternate chart that preserves the correspondence. One such choice of local co-ordinate is the complex stereographic projection. There are two different charts for two different localities :

$$\mathbb{C} = U_+ = S^2 - \{x_\infty\} \quad : \quad (x, y, z) \longleftrightarrow (r, Z_+, \bar{Z}_+) \quad \text{where} \quad Z_+ = \frac{x + iy}{r - z} \quad (3.1.17)$$

$$\bar{\mathbb{C}} = U_- = S^2 - \{x_0\} \quad : \quad (x, y, z) \longleftrightarrow (r, Z_-, \bar{Z}_-) \quad \text{where} \quad Z_- = \frac{x - iy}{r + z} \quad (3.1.18)$$

where the locality \mathbb{C} describes the entire sphere except for the north pole, while $\bar{\mathbb{C}}$ describes the same sphere, only this time exempting the south pole.

We can see that there are no arbitrary values in the appropriate localities:

$$U_- \quad : \quad (0, 0, \quad r) \longleftrightarrow (r, 0, 0)$$

$$U_+ \quad : \quad (0, 0, -r) \longleftrightarrow (r, 0, 0)$$

The correspondence to the polar co-ordinates is given by:

$$Z_+ = \frac{e^{i\phi} \sin \theta}{1 - \cos \theta}, \quad \bar{Z}_+ = \frac{e^{-i\phi} \sin \theta}{1 - \cos \theta} \quad Z_- = (Z_+)^{-1} \quad (3.1.19)$$

$$Z_+ \bar{Z}_- = e^{2i\phi} \quad (\bar{Z}_+)^{-1} Z_- = \tan \frac{\theta}{2} \quad (3.1.20)$$

However, to preserve the volume element under this diffeomorphism we need to obtain the appropriate vierbein. This can be done by adjusting the wedge product:

$$\begin{aligned} Z_+ &= \frac{e^{i\phi} \sin \theta}{1 - \cos \theta} = e^{i\phi} \cot \frac{\theta}{2} & \bar{Z}_+ &= \frac{e^{-i\phi} \sin \theta}{1 - \cos \theta} = e^{-i\phi} \cot \frac{\theta}{2} \\ dZ_+ &= \left(i \cot \frac{\theta}{2} d\phi - \frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2} d\theta \right) e^{i\phi} & d\bar{Z}_+ &= \left(-i \cot \frac{\theta}{2} d\phi - \frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2} d\theta \right) e^{-i\phi} \end{aligned}$$

$$\begin{aligned} \therefore dZ_+ \wedge d\bar{Z}_+ &= i \operatorname{cosec}^2 \frac{\theta}{2} \cot \frac{\theta}{2} d\theta \wedge d\phi = \frac{i}{2} \left(\frac{2}{1 - \cos \theta} \right)^2 \sin \theta d\theta \wedge d\phi \\ \Rightarrow -2i \left(\frac{1 - \cos \theta}{2} \right)^2 dZ_+ \wedge d\bar{Z}_+ &= \sin \theta d\theta \wedge d\phi \end{aligned}$$

$$|Z_+|^2 = \frac{\sin^2 \theta}{(1 - \cos \theta)^2} = \frac{1 + \cos \theta}{1 - \cos \theta} \quad \Rightarrow \quad 1 + |Z_+|^2 = \frac{2}{1 - \cos \theta}$$

$$\therefore \quad \xi_+ = -2i \frac{dZ_+ \wedge d\bar{Z}_+}{(1 + |Z_+|^2)^2} = \sin \theta d\theta \wedge d\phi \quad \omega = * \xi_+ + \xi_+ \quad (3.1.21)$$

This 2-form holds the same (form invariant) expression in the other locality as well :

$$dZ_+ = -(Z_-)^{-2} dZ_- \quad dZ_+ \wedge d\bar{Z}_+ = |Z_-|^{-4} dZ_- \wedge d\bar{Z}_- \quad (3.1.22)$$

$$1 + |Z_+|^2 = |Z_-|^{-2} (1 + |Z_-|^2) \quad (3.1.23)$$

$$\therefore \quad \xi_- = -2i \frac{dZ_- \wedge d\bar{Z}_-}{(1 + |Z_-|^2)^2} \quad \omega = * \xi_- + \xi_- \quad (3.1.24)$$

The respective volume element is given by:

$$\nu = \frac{1}{2} \omega \wedge \omega = -i \frac{r^2}{(1 + |Z_{\pm}|^2)^2} dt \wedge dr \wedge dZ_{\pm} \wedge d\bar{Z}_{\pm} \quad (3.1.25)$$

The closure of this 2-form term implies the existence of a potential field A , given by:

$$d\omega_{\pm} = 0 \quad \Rightarrow \quad \omega_{\pm} = dA_{\pm} \quad (3.1.26)$$

$$\omega_{\pm} = d \left(-\frac{r^3}{3} dt + i \frac{Z_{\pm} d\bar{Z}_{\pm} - \bar{Z}_{\pm} dZ_{\pm}}{1 + |Z_{\pm}|^2} \right) \quad \Rightarrow \quad A_{\pm} = -\frac{r^3}{3} dt + i \frac{Z_{\pm} d\bar{Z}_{\pm} - \bar{Z}_{\pm} dZ_{\pm}}{1 + |Z_{\pm}|^2} + d\varphi \quad (3.1.27)$$

Naturally, there is a chance of a constant or a first order exterior derivative separating the two potential form representations. To describe the connection between A_+ and A_- in the region $U_+ \cap U_-$, using (3.1.19), (3.1.22) and (3.1.23) we have the following results:

$$dA_+ = dA_- \quad \Rightarrow \quad A_+ = A_- + d\varphi \quad (3.1.28)$$

$$\begin{aligned}
A_+ &= -\frac{r^3}{3}dt + i\frac{Z_+d\bar{Z}_+ - \bar{Z}_+dZ_+}{1+|Z_+|^2} \\
Z_+d\bar{Z}_+ - \bar{Z}_+dZ_+ &= -\left(\frac{1}{Z_-}\frac{1}{(\bar{Z}_-)^2}d\bar{Z}_- - \frac{1}{\bar{Z}_-}\frac{1}{(Z_-)^2}dZ_-\right) = -\frac{Z_-d\bar{Z}_- - \bar{Z}_-dZ_-}{|Z_-|^4} \\
A_+ + \frac{r^3}{3}dt &= -\frac{1}{|Z_-|^2}\left(A_- + \frac{r^3}{3}dt\right) \quad \Rightarrow \quad A_+ - A_- = -i\frac{Z_-d\bar{Z}_- - \bar{Z}_-dZ_-}{|Z_-|^2} \quad (3.1.29)
\end{aligned}$$

Now, we can say that for a complex number:

$$\begin{aligned}
\frac{z d\bar{z} - \bar{z} dz}{|z|^2} &= -2i\frac{x dy - y dx}{x^2 + y^2} = -2i\frac{x^2}{x^2 + y^2}\frac{x dy - y dx}{x^2} \\
&= -2i\frac{1}{1 + \frac{y^2}{x^2}}d\left(\frac{y}{x}\right) = -2i d\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\} = -2i d(\arg(z))
\end{aligned}$$

Thus, we can say that:

$$A_+ - A_- = -2 d(\arg(Z_-)) = 2 d(\arg(Z_+))$$

$$\boxed{A_+ = A_- + 2 d(\arg(Z_+))} \quad (3.1.30)$$

Thus, as we can see that the two potentials for the two different localities, despite the same field strength form have a slight difference equivalent to the exterior derivative of the angular phase of the complex number.

Now we proceed to obtain the symplectic gauge fields associated with the metric and study its salient properties.

3.2 Symplectic Analysis

Using the Darboux co-ordinates, we can obtain a symplectic gauge field set (recall eq.(2.7)):

$$C_a = B_{ab}X^b \quad \theta^{ab} = \frac{1}{2}\eta_{ab}^3 \quad \Rightarrow \quad B_{ab} = -2\eta_{ab}^3 \quad (3.2.1)$$

$$\text{where} \quad \eta_{ab}^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (3.2.2)$$

In matrix form the set of symplectic gauge fields are

$$C = -2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \tau \\ \rho \\ x \\ y \end{pmatrix} = -2 \begin{pmatrix} \rho \\ -\tau \\ y \\ -x \end{pmatrix} = -2 \begin{pmatrix} \frac{1}{3}r^3 \\ -t \\ \phi \\ \cos \theta \end{pmatrix}$$

$$\therefore \quad C_1 = -\frac{2}{3}r^3, \quad C_2 = 2t, \quad C_3 = -2\phi \quad C_4 = -2\cos\theta \quad (3.2.3)$$

We can now derive the vector fields corresponding to the symplectic gauge fields (3.2.3) as the adjoint operation in the Poisson algebra and the result is shown in matrix form :

$$V_a(f) = \theta(C_a, f) \quad V_a^\mu = -\theta^{\mu\nu} \partial_\nu C_a \quad (3.2.4)$$

$$\therefore \quad V = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_t \\ \partial_r \\ \partial_\theta \\ \partial_\phi \end{pmatrix} \left(\frac{1}{3}r^3 - t \phi \cos\theta \right) = \begin{pmatrix} r^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin\theta \end{pmatrix} \quad (3.2.5)$$

We have the formula to relate the vector field with the vierbeins:

$$V_a = \lambda E_a \quad v^a = \lambda^{-1} e^a \quad (3.2.6)$$

To determine the value of λ , we make use of the relation:

$$\lambda^2 = \det V_a^\mu = r^2 \sin\theta \quad \Rightarrow \quad \lambda = r\sqrt{\sin\theta} \quad (3.2.7)$$

Now, the determinants of the volume preserving vector field array V_a^μ and that of the inverse vector field array, or corresponding vierbein array are given by:

$$\det(V_a^\mu) = \begin{vmatrix} r^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin\theta \end{vmatrix} = r^2 \sin\theta \quad \det(V_\mu^a) = \begin{vmatrix} \frac{1}{r^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sin\theta} \end{vmatrix} = \frac{1}{r^2 \sin\theta} \quad (3.2.8)$$

Knowing that $\lambda^2 = r^2 \sin\theta$ we can say that:

$$\det(V_\mu^a) = \frac{1}{\lambda^2} \quad \Rightarrow \quad \lambda^2 = \frac{1}{\det(V_\mu^a)} \quad \Rightarrow \quad v(x) = 1 \quad (3.2.9)$$

thus concluding that the inverse vierbein fields satisfy equation (5.145) of [16].

3.3 Bianchi identity for Symplectic gauge and Vector fields

The Jacobi and Bianchi identities are well-studied in differential geometry. One can even suspect that both are derivatives of a basic identity defined by:

$$d^2\omega^n = 0 \quad (3.3.1)$$

where ω^n is an n-form. If such speculation is well founded, then having arisen from the same source, one can suppose a connection between the two identities. Such a suspicion is further strengthened from the apparent similarity they have in their appearance at first glance. Here, we will attempt to put this suspicion to rest.

We begin by considering the wedge product of the first order exterior derivatives of any two arbitrary functions f and g

$$\begin{aligned} df \wedge dg &= \partial_\mu f \partial_\nu g (dx^\mu \wedge dx^\nu) \\ &= \partial_\mu f \partial_\nu g (dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu) \\ &= (\partial_\mu f \partial_\nu g - \partial_\nu f \partial_\mu g) dx^\mu \otimes dx^\nu \equiv \{f, g\} \end{aligned}$$

where

$$\{f, g\}_\theta = \theta(df \wedge dg) = \theta(df, dg)$$

Then we take their mixed 2nd order exterior derivative and some simple manipulations:

$$\begin{aligned} df \wedge d(dg \wedge dh) &= -d(df \wedge dg \wedge dh) \quad (= 0) \\ &= -\partial_\mu (\partial_\nu f \partial_\rho g \partial_\sigma h) dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \\ &= -\frac{1}{3} \partial_\mu (\partial_\nu f \partial_\rho g \partial_\sigma h) dx^\mu \wedge (dx^\nu \wedge dx^\rho \wedge dx^\sigma + dx^\rho \wedge dx^\sigma \wedge dx^\nu + dx^\sigma \wedge dx^\nu \wedge dx^\rho) \\ &= -\frac{1}{3} \partial_\mu (\partial_\nu f \partial_\rho g \partial_\sigma h + \partial_\nu g \partial_\rho h \partial_\sigma f + \partial_\nu h \partial_\rho f \partial_\sigma g) dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \\ &= \frac{1}{3} \{ \partial_\nu f \partial_\mu (\partial_\rho g \partial_\sigma h) + \partial_\nu g \partial_\mu (\partial_\rho h \partial_\sigma f) + \partial_\nu h \partial_\mu (\partial_\rho f \partial_\sigma g) \} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \\ &= \frac{1}{3} [\{f, \{g, h\}\}_\theta + \{g, \{h, f\}\}_\theta + \{h, \{f, g\}\}_\theta] \end{aligned}$$

We now arrive at this important identity,

$$\therefore \{f, \{g, h\}\}_\theta + \{g, \{h, f\}\}_\theta + \{h, \{f, g\}\}_\theta = 0 \quad (3.3.2)$$

We realize that to some extent, the Jacobi identity (3.3.2) is connected to the identity (3.3.1). We can proceed to further verify it for our case. According to (3.2.4), the various Poisson Brackets between the gauge fields (3.2.3) are computed to be:

$$\{C_a, C_b\}_\theta = \theta(C_a, C_b) = V_a(C_b) = -V_b(C_a) \quad (3.3.3)$$

$$\{C_a, \{C_b, C_c\}_\theta\}_\theta = V_a(V_b(C_c)) \quad (3.3.4)$$

$$\begin{aligned} \{C_1, C_2\}_\theta &= 2r^2, & \{C_1, C_3\}_\theta &= 0, & \{C_1, C_4\}_\theta &= 0 \\ \{C_2, C_3\}_\theta &= 0, & \{C_2, C_4\}_\theta &= 0, & \{C_3, C_4\}_\theta &= 2 \sin \theta \end{aligned}$$

$$\begin{aligned} \{C_1, \{C_2, C_3\}_\theta\}_\theta &= 0, & \{C_1, \{C_2, C_4\}_\theta\}_\theta &= 0 & \{C_1, \{C_3, C_4\}_\theta\}_\theta &= 0 \\ \{C_2, \{C_1, C_3\}_\theta\}_\theta &= 0, & \{C_2, \{C_1, C_4\}_\theta\}_\theta &= 0 & \{C_2, \{C_3, C_4\}_\theta\}_\theta &= 0 \\ \{C_3, \{C_1, C_2\}_\theta\}_\theta &= 0, & \{C_3, \{C_1, C_4\}_\theta\}_\theta &= 0 & \{C_3, \{C_2, C_4\}_\theta\}_\theta &= 0 \\ \{C_4, \{C_1, C_2\}_\theta\}_\theta &= 0, & \{C_4, \{C_1, C_3\}_\theta\}_\theta &= 0 & \{C_4, \{C_2, C_3\}_\theta\}_\theta &= 0 \end{aligned}$$

Thus, we see whichever way around we take it, the Jacobi identity is always satisfied.

$$\{C_{[a}, \{C_b, C_c\}_\theta\}_\theta = 0 \quad (3.3.5)$$

Also, we can see that this translates to an analogous identity for vector fields as is clear from the following simple identification and few steps that follows subsequently:

$$V_a = \{C_a, \cdot\}_\theta \quad (3.3.6)$$

$$\begin{aligned} [V_a, V_b] &= V_a V_b - V_b V_a = \{C_a, \{C_b, \cdot\}_\theta\}_\theta - \{C_b, \{C_a, \cdot\}_\theta\}_\theta \\ &= -\{C_b, \{\cdot, C_a\}_\theta\}_\theta - \{\cdot, \{C_a, C_b\}_\theta\}_\theta - \{C_b, \{C_a, \cdot\}_\theta\}_\theta \\ \Rightarrow [V_a, V_b] &= \{\{C_a, C_b\}_\theta, \cdot\}_\theta \end{aligned} \quad (3.3.7)$$

$$\begin{aligned} [V_a, [V_b, V_c]] &= \{C_a, \{\{C_b, C_c\}_\theta, \cdot\}_\theta\}_\theta - \{\{C_b, C_c\}_\theta, \{C_a, \cdot\}_\theta\}_\theta \\ &= -\{\{C_b, C_c\}_\theta, \{\cdot, C_a\}_\theta\}_\theta - \{\cdot, \{C_a, \{C_b, C_c\}_\theta\}_\theta\}_\theta - \{\{C_b, C_c\}_\theta, \{C_a, \cdot\}_\theta\}_\theta \\ \Rightarrow [V_a, [V_b, V_c]] &= \{\{C_a, \{C_b, C_c\}_\theta\}_\theta, \cdot\}_\theta \end{aligned} \quad (3.3.8)$$

$$\therefore [V_a, [V_b, V_c]] + [V_b, [V_c, V_a]] + [V_c, [V_a, V_b]] = 0 \quad (3.3.9)$$

known as the Bianchi identity for vector fields.

Thus, we can conclude that there exists a connection (which has its roots at the important isomorphism $\mathfrak{H} : \mathfrak{P}(M) \rightarrow \mathfrak{X}(M)$ between the set of symplectic gauge fields that form the Poisson algebra and the vector fields forming Lie algebra) between the two identities which is described as shown.

$$\begin{aligned} &\{C_a, \{C_b, C_c\}_\theta\}_\theta + \{C_b, \{C_c, C_a\}_\theta\}_\theta + \{C_c, \{C_a, C_b\}_\theta\}_\theta = 0 \\ &\quad \Updownarrow \\ &[V_a, [V_b, V_c]] + [V_b, [V_c, V_a]] + [V_c, [V_a, V_b]] = 0 \end{aligned}$$

However, the above identities are valid only in regions where the metric is properly defined. They break down in the presence of singularity as evident in electrodynamics where we find the Bianchi identity invalid in the presence of static and dynamic charge (current) distributions.

$$A = A_i dx^i \quad A_i = \{\varphi, \vec{A}\} \quad F = dA \quad F_{ij} = \partial_i A_j - \partial_j A_i \equiv \{\vec{E}, \vec{B}\}$$

$$\begin{aligned} \{\rho, \vec{J}\} = \{0, \vec{0}\} : \quad dF = 0 &\longrightarrow \vec{\nabla} \cdot \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0 \\ \{\rho, \vec{J}\} \neq \{0, \vec{0}\} : \quad dF \neq 0 &\longrightarrow \vec{\nabla} \cdot \vec{E} = \rho, \quad \vec{\nabla} \cdot \vec{B} = 0 \end{aligned}$$

Since the Schwarzschild space has an irremovable singularity at the origin, the Bianchi identity is invalid there. Also, we can note that from before in (3.3.5) and (3.3.8), and remembering (2.24), we can conclude that:

$$[V_a, [V_b, V_c]] = 0 \quad \Rightarrow \quad R_{abc}{}^d = 0 \quad (3.3.10)$$

Showing that the local results are more or less consistent with our theory.

3.4 Seiberg Witten map and absence of self-duality

Seiberg and Witten showed [17] that there are two equivalent descriptions - comutative and non-commutative of the low energy effective theory, depending on the regularization scheme or path integral prescription for the open string ending on a D-brane.

Since these two descriptions arise from the same open string theory depending on different regularizations, and the physics being independent of the regularization scheme, Seiberg and Witten argued that they should be equivalent. Thus there must be a spacetime field redefinition between ordinary and NC gauge fields, so called the Seiberg-Witten (SW) map.

The relation for the field strength \hat{F} is given by (see eqn.(20) of [13]):

$$\{C_a, C_b\}_\theta = -B_{ab} + \hat{F}_{ab} \quad \Rightarrow \quad \hat{F}_{ab} = B_{ab} + \{C_a, C_b\}_\theta \quad (3.4.1)$$

Using the C matrix from (3.2.3), we can write:

$$\{C_a, C_b\}_\theta = \begin{pmatrix} 0 & 2r^2 & 0 & 0 \\ -2r^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sin\theta \\ 0 & 0 & -2\sin\theta & 0 \end{pmatrix}$$

$$\hat{F} = -2 \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2r^2 & 0 & 0 \\ -2r^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sin\theta \\ 0 & 0 & -2\sin\theta & 0 \end{pmatrix} \quad (3.4.2)$$

$$\therefore \hat{F} = -2 \begin{pmatrix} 0 & 1-r^2 & 0 & 0 \\ -(1-r^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\sin\theta \\ 0 & 0 & -(1-\sin\theta) & 0 \end{pmatrix} \quad (3.4.3)$$

At this point, we recapitulate the Seiberg-Witten map between the field strengths of the two descriptions - commutative and non-commutative, given by the formula:

$$\hat{F} = (1 + F\theta)^{-1} F \quad \Rightarrow \quad F = \hat{F}(1 - \theta\hat{F})^{-1} \quad (3.4.4)$$

$$(1 - \theta\hat{F})^{-1} = \begin{pmatrix} r^2 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & \sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{r^2} & 0 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\sin\theta} & 0 \\ 0 & 0 & 0 & \frac{1}{\sin\theta} \end{pmatrix}$$

It is easy to see that the commutative gauge field strength $F_{\mu\nu}$ shows no duality at all.

$$F = -2 \begin{pmatrix} 0 & \frac{1-r^2}{r^2} & 0 & 0 \\ -\frac{1-r^2}{r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\sin\theta}{\sin\theta} \\ 0 & 0 & -\frac{1-\sin\theta}{\sin\theta} & 0 \end{pmatrix} \quad (3.4.5)$$

3.5 The corresponding equation of motion

Now we consider the equation of motion of the gauge fields (3.4.5). We start by looking at the action corresponding to the gauge fields:

$$S = \frac{1}{4g_{YM}} \int d^4y \{C_a, C_b\}^2 \quad (3.5.1)$$

$$\begin{aligned} \hat{F} - B &= (1 + F\theta)^{-1} F - B \\ &= (1 + F\theta)^{-1} \{F - (1 + F\theta) \cdot B\} \\ &= (1 + F\theta)^{-1} \{F - B - F\} = -G^{-1} B \end{aligned}$$

where we have chosen to substitute $G = 1 + F\theta = \begin{pmatrix} \frac{1}{r^2} & 0 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\sin\theta} & 0 \\ 0 & 0 & 0 & \frac{1}{\sin\theta} \end{pmatrix}$.

$$\therefore S = \int d^4y \{C_\mu, C_\nu\}_\theta^2 = \int d^4x \sqrt{\text{Det}(G)} G^{\mu\lambda} G^{\nu\gamma} B_{\lambda\nu} B_{\gamma\mu} \quad (3.5.2)$$

The equation of motion can be obtained by minimising the action variation:

$$\begin{aligned} \int d^4y \text{Tr}(\hat{F} - B)^2 &= \int d^4x \sqrt{\text{Det}(G)} \text{Tr}(\hat{F} - B)^2 \\ \int d^4y (\hat{F} - B)^{\mu\nu} (\hat{F} - B)_{\mu\nu} &= \int d^4x \sqrt{\text{Det}(G)} (\hat{F} - B)^{\mu\nu} (\hat{F} - B)_{\mu\nu} \\ &= \int d^4x \sqrt{\text{Det}(G)} (G^\mu{}_\lambda B^{\lambda\nu}) (B_{\mu\gamma} G^\gamma{}_\nu) \\ (G^\mu{}_\lambda B^{\lambda\nu}) (B_{\mu\gamma} G^\gamma{}_\nu) &= (G^{\mu\lambda} B_\lambda{}^\nu) (B_{\mu\gamma} G^\gamma{}_\nu) = (G^{\mu\lambda} B_{\lambda\nu}) (B_{\mu\gamma} G^{\gamma\nu}) \\ &= G^{\mu\lambda} G^{\nu\gamma} B_{\mu\gamma} B_{\lambda\nu} \\ \therefore \int d^4y (\hat{F} - B)^{\mu\nu} (\hat{F} - B)_{\mu\nu} &= \int d^4x \sqrt{\text{Det}(G)} G^{\mu\lambda} G^{\nu\gamma} B_{\mu\gamma} B_{\lambda\nu} \\ &= - \int d^4x \sqrt{\text{Det}(G)} \text{Tr}(G^{-1} B G^{-1} B) \end{aligned} \quad (3.5.3)$$

Noting that $A^{[\mu\nu]} = \frac{1}{2}(A^{\mu\nu} - A^{\nu\mu})$, the commutative equation of motion is derived as:

$$\begin{aligned} \delta S = 0 &\Rightarrow \delta \left[\int d^4x \sqrt{\text{Det}(G)} \text{Tr}(G^{-1} B G^{-1} B) \right] = 0 \\ \Rightarrow \int d^4x [\delta(\sqrt{\text{Det}(G)}) \text{Tr}(G^{-1} B G^{-1} B) + \sqrt{\text{Det}(G)} \cdot \delta\{\text{Tr}(G^{-1} B G^{-1} B)\}] &= 0 \end{aligned}$$

In operator form, we write:

$$\delta \sqrt{\text{Det}(G)} = \frac{1}{2} \sqrt{\text{Det}(G)} G^{-1} \delta(G) = \frac{1}{2} \sqrt{\text{Det}(G)} (G^{-1}) \theta \delta F$$

$$\begin{aligned}
G^{-1}.G = \mathbb{I} &\Rightarrow \delta((G^{-1})).G = -G^{-1}.\delta(G) = -G^{-1}.\theta.\delta F \\
&\Rightarrow \delta((G^{-1})) = -(\theta.G^{-1}).\delta F.G^{-1}
\end{aligned}$$

Thus, the minimised action variation is:

$$\begin{aligned}
\therefore \int d^4x \sqrt{\text{Det}(G)} &\left[(G^{-1})\theta.\text{Tr}(G^{-1}BG^{-1}B)\delta F + 4\text{Tr}(G^{-1}B\delta(G^{-1})B) \right] = 0 \\
&\Rightarrow \int d^4x \sqrt{\text{Det}(G)} \left[(\theta.G^{-1})\text{Tr}(G^{-1}BG^{-1}B) + 4(G^{-1}B(\theta.G^{-1})BG^{-1}) \right]^{\mu\nu} \delta F_{\mu\nu} = 0
\end{aligned}$$

The variation of the gauge field F and its application into the action variation are:

$$\delta F_{\mu\nu} = \delta(\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu$$

$$\begin{aligned}
\therefore \int d^4x \sqrt{\text{Det}(G)} &\left[(\theta.G^{-1})\text{Tr}(G^{-1}BG^{-1}B) - 4(G^{-1}B(\theta.G^{-1})BG^{-1}) \right]^{\mu\nu} \partial_\mu \delta A_\nu = 0 \\
&\Rightarrow \partial_\mu \left[\sqrt{G} \{ (\theta G^{-1})^{\mu\nu} \text{Tr}(G^{-1}BG^{-1}B) - 4(\theta G^{-1}BG^{-1}BG^{-1})^{[\mu\nu]} \} \right] = 0
\end{aligned}$$

Thus, the resulting equation of motion is:

$$\boxed{\partial_\mu \left[\sqrt{G} \{ (\theta G^{-1})^{\mu\nu} \text{Tr}(G^{-1}BG^{-1}B) - 4(\theta G^{-1}BG^{-1}BG^{-1})^{[\mu\nu]} \} \right] = 0} \quad (3.5.4)$$

4. Geometric Analysis

Now we proceed to analyze the various geometric and topological properties of the Euclidean Schwarzschild metric. This will involve obtaining the various topological invariants related to the metric. We will start by obtaining the curvature components of the metric.

We can extract the complete set of vierbeins for the metric (3.1) as:

$$\begin{aligned}
e^1 &= \sqrt{1 - \frac{2m}{r}} dt & e^2 &= \frac{1}{\sqrt{1 - \frac{2m}{r}}} dr \\
e^3 &= r d\theta & e^4 &= r \sin \theta d\varphi
\end{aligned} \quad (4.1)$$

Starting with the vierbeins in (4.1) and using Cartan's 1st structure equation for torsion free case, we have:

$$\begin{aligned}
de^1 &= \frac{m}{r^2 \sqrt{1 - \frac{2m}{r}}} dr \wedge dt = -\frac{m}{r^2} dt \wedge \frac{dr}{\sqrt{1 - \frac{2m}{r}}} \\
&\Rightarrow \omega^1_2 = -\omega^2_1 = \frac{m}{r^2} dt
\end{aligned} \quad (4.2)$$

$$\begin{aligned}
de^3 &= dr \wedge d\theta = -\sqrt{1 - \frac{2m}{r}} d\theta \wedge \frac{dr}{\sqrt{1 - \frac{2m}{r}}} \\
&\Rightarrow \omega^3_2 = -\omega^2_3 = \sqrt{1 - \frac{2m}{r}} d\theta
\end{aligned} \quad (4.3)$$

$$\begin{aligned}
de^4 &= \sin \theta \, dr \wedge d\varphi + r \cos \theta \, d\theta \wedge d\varphi \\
&= -\sqrt{1 - \frac{2m}{r}} \sin \theta \, d\varphi \wedge \frac{dr}{\sqrt{1 - \frac{2m}{r}}} - \cos \theta \, d\varphi \wedge r d\theta \\
\Rightarrow \quad \omega^4_2 &= -\omega^2_4 = \sqrt{1 - \frac{2m}{r}} \sin \theta \, d\varphi
\end{aligned} \tag{4.4}$$

$$\omega^4_3 = -\omega^3_4 = \cos \theta \, d\varphi \tag{4.5}$$

The overall ω (spin-connection) matrix is given by:

$$\omega^i_j = \frac{1}{r} \begin{pmatrix} 0 & a e^1 & 0 & 0 \\ -a e^1 & 0 & -b e^3 & -b e^4 \\ 0 & b e^3 & 0 & -c e^4 \\ 0 & b e^4 & c e^4 & 0 \end{pmatrix} \tag{4.6}$$

$$\text{where} \quad a = \frac{m}{r\sqrt{f(r)}} \quad b = \sqrt{f(r)} \quad c = \cot \theta \tag{4.7}$$

For the curvature components, we use the 2nd structure equation:

$$R^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j \tag{4.8}$$

Combining (4.6) and (4.8) gives us the following non-vanishing components for the Riemann tensor:

$$\begin{aligned}
d\omega^1_2 &= -d\omega^2_1 = -\frac{2m}{r^3} dr \wedge dt = \frac{2m}{r^3} \sqrt{1 - \frac{2m}{r}} dt \wedge \frac{dr}{\sqrt{1 - \frac{2m}{r}}} \\
\Rightarrow \quad &\boxed{R^1_{212} = -R^1_{221} = -R^2_{112} = R^2_{121} = \frac{2m}{r^3}}
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
\omega^1_2 \wedge \omega^2_3 &= -(\omega^3_2 \wedge \omega^2_1) = -\frac{m}{r^2} \sqrt{1 - \frac{2m}{r}} dt \wedge d\theta = -\frac{m}{r^3} \sqrt{1 - \frac{2m}{r}} dt \wedge r d\theta \\
\Rightarrow \quad &\boxed{R^1_{313} = -R^1_{331} = -R^3_{113} = R^3_{131} = -\frac{m}{r^3}}
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
\omega^1_2 \wedge \omega^2_4 &= -(\omega^4_2 \wedge \omega^2_1) = -\frac{m}{r^2} \sqrt{1 - \frac{2m}{r}} \sin \theta dt \wedge d\varphi = -\frac{m}{r^3} \sqrt{1 - \frac{2m}{r}} dt \wedge r \sin \theta d\varphi \\
\Rightarrow \quad &\boxed{R^1_{414} = -R^1_{441} = -R^4_{114} = R^4_{141} = -\frac{m}{r^3}}
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
d\omega^2_3 &= -d\omega^3_2 = -\frac{m}{r^2 \sqrt{1 - \frac{2m}{r}}} dr \wedge d\theta = -\frac{m}{r^3} \frac{dr}{\sqrt{1 - \frac{2m}{r}}} \wedge r d\theta \\
\Rightarrow \quad &\boxed{R^2_{323} = -R^2_{332} = -R^3_{223} = R^3_{232} = -\frac{m}{r^3}}
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
d\omega^2_4 + \omega^2_3 \wedge \omega^3_4 &= -(d\omega^4_2 + \omega^4_3 \wedge \omega^3_2) \\
&= -\frac{m}{r^2 \sqrt{1 - \frac{2m}{r}}} \sin \theta dr \wedge d\varphi = -\frac{m}{r^3} \frac{dr}{\sqrt{1 - \frac{2m}{r}}} \wedge r \sin \theta d\varphi \\
\Rightarrow \quad &\boxed{R^2_{424} = -R^2_{442} = -R^4_{224} = R^4_{242} = -\frac{m}{r^3}} \tag{4.13}
\end{aligned}$$

$$\begin{aligned}
d\omega^3_4 + \omega^3_2 \wedge \omega^2_4 &= -(d\omega^4_3 + \omega^4_2 \wedge \omega^2_3) \\
&= \frac{2m}{r} \sin \theta d\theta \wedge d\varphi = \frac{2m}{r^3} r d\theta \wedge r \sin \theta d\varphi \\
\Rightarrow \quad &\boxed{R^3_{434} = -R^3_{443} = -R^4_{334} = R^4_{343} = \frac{2m}{r^3}} \tag{4.14}
\end{aligned}$$

In a compact form, the R_{ab} matrix can be written as:

$$R_{ab} = \frac{m}{r^3} \begin{pmatrix} 0 & 2x & -y & -z \\ -2x & 0 & -\bar{z} & \bar{y} \\ y & \bar{z} & 0 & 2\bar{x} \\ z & -\bar{y} & -2\bar{x} & 0 \end{pmatrix} \tag{4.15}$$

where we use the representation:

$$\begin{aligned}
x &= e^1 \wedge e^2 & y &= e^1 \wedge e^3 & z &= e^1 \wedge e^4 \\
\bar{x} &= e^3 \wedge e^4 & \bar{y} &= e^4 \wedge e^2 & \bar{z} &= e^2 \wedge e^3
\end{aligned} \tag{4.16}$$

with the following condition

$$x \wedge \bar{x} = y \wedge \bar{y} = z \wedge \bar{z} = \nu \tag{4.17}$$

where ν is the volume form.

Clearly, we can see that R_{ab} matrix of (4.15) is not self dual since each of its components are made of only one 2-form term, making it impossible to exhibit self-duality. ie.:-

$${}^*R_{ab} = \frac{1}{2} \frac{\varepsilon_{ab}{}^{cd}}{\sqrt{g}} R_{cd} \neq R_{ab} \tag{4.18}$$

Now with the Riemann tensor components, we can compute the Ricci tensor and Ricci scalar which are all vanishing.

$$R_{ij} = \eta^{kl} R_{ikjl} = \eta_{im} \eta^{kl} R^m{}_{kjl} \tag{4.19}$$

$$\begin{aligned}
R_{11} &= R_{1212} + R_{1313} + R_{1414} = 0 \\
R_{22} &= R_{2121} + R_{2323} + R_{2424} = 0 \\
R_{33} &= R_{3131} + R_{3232} + R_{3434} = 0 \\
R_{44} &= R_{4141} + R_{4242} + R_{4343} = 0
\end{aligned} \tag{4.20}$$

$$R = \eta^{ij} R_{ij} \quad (4.21)$$

$$\therefore R = R_{11} + R_{22} + R_{33} + R_{44} = 0 \quad (4.22)$$

So the Euclidean Schwarzschild solution is not a gravitational instanton (not a hyperkahler manifold) although it is a Ricci-flat manifold. Since the spin connections in eq. (4.6) are neither self-dual or anti-self dual, we can proceed to construct both type of SU(2) gauge fields and the field strengths using respectively the spin connections (4.6) and curvature components (4.15) using the following formula:

$$A^{(\pm)i} = \frac{1}{4} \eta_{\mu\nu}^{(\pm)i} \omega_{\mu\nu} \quad F^{(\pm)i} = \frac{1}{4} \eta_{\mu\nu}^{(\pm)i} R_{\mu\nu} \quad (4.23)$$

By construction the field strengths should be either self-dual (for the + sign) or anti-self dual (for the - sign). According to a general result (3.41) found in [10], the SU(2) gauge field (4.23) automatically satisfy the self duality equation and hence these solution describes an SU(2) Yang-Mills (anti) instanton on the space (3.1).

Thus, we have the following description for the $SU(2)_+$ instanton and $SU(2)_-$ anti-instanton gauge fields respectively listed as :

$$\begin{aligned} A^{(+1)} &= -\frac{1}{2r} b e^3 & A^{(-1)} &= -\frac{b}{2r} e^3 \\ A^{(+2)} &= -\frac{1}{2r} b e^4 & A^{(-2)} &= -\frac{b}{2r} e^4 \\ A^{(+3)} &= \frac{1}{2r} (a e^1 - c e^4) & A^{(-3)} &= \frac{1}{2r} (a e^1 + c e^4) \end{aligned} \quad (4.24)$$

$$\begin{aligned} F^{(+1)} &= -\frac{m}{2r^3} (z + \bar{z}) & F^{(-1)} &= \frac{m}{2r^3} (z - \bar{z}) \\ F^{(+2)} &= \frac{m}{2r^3} (y + \bar{y}) & F^{(-2)} &= \frac{m}{2r^3} (y - \bar{y}) \\ F^{(+3)} &= \frac{m}{r^3} (x + \bar{x}) & F^{(-3)} &= \frac{m}{r^3} (x - \bar{x}) \end{aligned} \quad (4.25)$$

Remembering that the curvature components are given by (4.8), we can write:

$$\begin{aligned} R^a_b &= \frac{1}{2} R^a_{b\mu\nu} dx^\mu \wedge dx^\nu \quad \Rightarrow \quad R^a_{bcd} = \iota_{E_d} \iota_{E_c} R^a_b \\ d\omega^a_b + \omega^a_m \wedge \omega^m_b &= (\partial_\mu \omega_\nu^a_b + \omega_\mu^a_m \omega_\nu^m_b) dx^\mu \wedge dx^\nu \\ \iota_{E_d} \iota_{E_c} (d\omega^a_b + \omega^a_m \wedge \omega^m_b) &= (E_d^\nu \partial_c \omega_\nu^a_b + \omega_c^a_m \omega_d^m_b) \\ &= \{ \partial_c (E_d^\nu \omega_\nu^a_b) - \partial_c (E_d^\nu) \omega_\nu^a_b + \omega_c^a_m \omega_d^m_b \} \\ &= \{ \partial_c (\omega_d^a_b) - \omega_c^m_d (E_m^\nu \omega_\nu^a_b) + \omega_c^a_m \omega_d^m_b \} \\ &= \{ \partial_c (\omega_d^a_b) - \omega_c^m_d \omega_m^a_b + \omega_c^a_m \omega_d^m_b \} \\ \therefore R^a_{bcd} &= \{ \nabla_c (\omega_d^a_b) - \omega_c^m_d \omega_m^a_b + \omega_c^a_m \omega_d^m_b \} \end{aligned} \quad (4.26)$$

Thus we get the Ricci tensor to be

$$\begin{aligned} R^a{}_c &= \{ \nabla_c(\omega_b^a{}_b) - \omega_c^m{}_b \omega_m^a{}_b + \omega_c^a{}_m \omega_b^m{}_b \} \\ \Rightarrow R_{ac} &= \{ \underbrace{\nabla_c(f_{bab})}_{\parallel} - \omega_{cmb} \omega_{mab} - \omega_{cma} f_{bmb} \} \\ &= 0 \end{aligned}$$

Finally the Ricci scalar can be written as

$$R = -\{ \omega_{amb} \omega_{mab} + f_{ama} f_{bmb} \} \quad (4.27)$$

Now, since the Ricci scalar vanishes in our case (see eqn. (4.22)), we have:

$$R = 0 \quad \Rightarrow \quad (f_{aba})^2 = \omega_{abc} \omega_{cab} \quad (4.28)$$

The vierbeins and the vector fields in (3.2.4)-(3.2.6) exhibit the structure equations:

$$[E_a, E_b] = -f_{ab}^c E_c \quad [V_a, V_b] = -g_{ab}^c V_c \quad (4.29)$$

If the vector fields $\{E_a\}$ and $\{V_a\}$ are related by (3.2.6), then we can suppose that:

$$\begin{aligned} dV_b &= d(\lambda E_b) = d\lambda \wedge E_b - \lambda \omega_b^c E_c \\ &= d\lambda \wedge E_b - \omega_b^c V_c \\ &= d(\log \lambda) \wedge V_b - \omega_b^c V_c \end{aligned}$$

$$\iota_{V_a} dV_b = V_a V_b = V_a(\log \lambda) V_b - \lambda \omega_a^c{}_b V_c$$

$$\begin{aligned} \therefore [V_a, V_b] &= V_a(\log \lambda) V_b - V_b(\log \lambda) V_a - \lambda(\omega_a^c{}_b - \omega_b^c{}_a) V_c \\ \Rightarrow -g_{ab}^c V_c &= \frac{1}{2}(g_{ma}^m V_b - g_{mb}^m V_a) - \lambda f_{ab}^c V_c \end{aligned}$$

So we can write the structure constants in terms of the metric

$$\begin{aligned} f_{ab}^c &= \frac{1}{\lambda} \left\{ g_{ab}^c + \frac{1}{2} \left(g_{ma}^m \delta_b^c - g_{mb}^m \delta_a^c \right) \right\} \\ \Rightarrow f_{ab}^a &= \frac{1}{\lambda} \left\{ g_{ab}^a + \frac{1}{2} \left(g_{ma}^m \delta_b^a - g_{mb}^m \delta_a^a \right) \right\} = \frac{1}{\lambda} g_{ab}^a \end{aligned} \quad (4.30)$$

We also note the relation between spin connection and structure constant:

$$\omega_{abc} = \frac{1}{2} (f_{abc} - f_{bca} + f_{cab}) \quad (4.31)$$

Keeping in mind that the Levi-Civita symbol exhibits the following relation:

$$\begin{aligned} \varepsilon^{abcd} &= -\varepsilon^{bcda} = \varepsilon^{cdab} = \alpha = \pm 1 \\ \Rightarrow \alpha \varepsilon^{abcd} &= -\alpha \varepsilon^{bcda} = \alpha \varepsilon^{cdab} = 1 \\ \Rightarrow \alpha \varepsilon^{abcd} &= 1, \quad \alpha \varepsilon^{bcda} = -1, \quad \alpha \varepsilon^{cdab} = 1 \end{aligned} \quad (4.32)$$

we can use this property to re-write (4.31) as:

$$\begin{aligned}
\omega_{abc} &\equiv \frac{\alpha}{2} \left(\varepsilon^{abcd} f_{abc} + \varepsilon^{bcd a} f_{bca} + \varepsilon^{cdab} f_{cab} \right) \\
&= \frac{\alpha}{2} \left(\varepsilon^{abcd} f_{abc} - \varepsilon^{bcad} f_{bca} + \varepsilon^{cabd} f_{cab} \right) \\
&= \frac{\alpha}{2} \left(\varepsilon^{abcd} - \varepsilon^{abcd} + \varepsilon^{abcd} \right) f_{abc} = \frac{\alpha}{2} \varepsilon^{abcd} f_{abc} \\
\therefore \quad \omega_{abc} \omega_{cab} &\equiv \alpha^2 \left(\frac{1}{2} \varepsilon^{abcd} f_{abc} \right)^2 = \left(\frac{1}{2} \varepsilon^{abcd} f_{abc} \right)^2
\end{aligned} \tag{4.33}$$

Applying (4.30) to the above result, we obtain:

$$\varepsilon^{abcd} f_{abc} = \frac{\varepsilon^{abcd}}{\lambda} \left\{ g_{abc} + \frac{1}{2} \left(g_{ma}^m \delta_b^c - g_{mb}^m \delta_a^c \right) \right\} = \frac{\varepsilon^{abcd}}{\lambda} g_{abc} \tag{4.34}$$

Thus, using (4.28) and (4.30) together we have:

$$\begin{aligned}
(f_{aba})^2 &= \left(\frac{1}{2} \varepsilon^{abcd} f_{abc} \right)^2 \Rightarrow \left(\frac{1}{\lambda} g_{ab}^a \right)^2 = \frac{1}{\lambda^2} \left(\frac{1}{2} \varepsilon^{abcd} g_{abc} \right)^2 \\
&\Rightarrow (g_{ab}^a)^2 = \left(\frac{1}{2} \varepsilon^{abcd} g_{abc} \right)^2
\end{aligned}$$

Finally we will make use of the following relations to prove an identity here:

$$\rho^b = g_a^{ba} \quad \Psi^d = \frac{1}{2} \varepsilon^{abcd} g_{abc} \tag{4.35}$$

$$\rho_b \rho^b = \Psi_d \Psi^d \Rightarrow \rho^a = \pm \Psi^a \tag{4.36}$$

With a little effort, it can be shown (in any $2n$ -dimensions) [6, 15] that the right-hand side of (3.3.9) is precisely equivalent to the first Bianchi identity of Riemann curvature tensors, i.e.,

$$[V_a, [V_b, V_c]] + \text{cyclic} = 0 \quad \Leftrightarrow \quad R_{[abc]d} = 0, \tag{4.37}$$

where $[abc]$ denotes the cyclic permutation of indices. The equation (4.37) leads to a cryptic result for Ricci tensors [6, 15]

$$R_{ab} = -\frac{1}{\lambda^2} \left[g_d^{(+i)} g_d^{(-j)} \left(\eta_{ac}^i \bar{\eta}_{bc}^j + \eta_{bc}^i \bar{\eta}_{ac}^j \right) - g_c^{(+i)} g_d^{(-j)} \left(\eta_{ac}^i \bar{\eta}_{bd}^j + \eta_{bd}^i \bar{\eta}_{ac}^j \right) \right] \tag{4.38}$$

where η_{ab}^i and $\bar{\eta}_{ab}^i$ are self-dual and anti-self-dual 't Hooft symbols. To get the result (4.38), we have to define the canonical decomposition of the structure equation (4.29) like

$$g_{abc} = g_c^{(+i)} \eta_{ab}^i + g_c^{(-i)} \bar{\eta}_{ab}^i. \tag{4.39}$$

A notable point is that the right-hand side of (4.38) consists of purely interaction terms between self-dual and anti-self-dual parts in (4.39) which is the feature withheld by matter fields only [11, 18]. A gravitational instanton which is a Ricci-flat, Kähler manifold can be

understood as either $g_c^{(-)i} = 0$ (self-dual) or $g_c^{(+)i} = 0$ (anti-self-dual) in terms of (4.39) and so $R_{ab} = 0$ in (4.38). Hence, the result (4.38) is consistent with the Ricci-flatness of gravitational instantons. However (4.38) also has a nontrivial trace contribution, i.e., a nonzero Ricci scalar, due to the second part which does not exist in Einstein gravity as was shown in [18]. The content of the energy-momentum tensor defined by the right-hand side of (3.3.9) becomes manifest by decomposing it into two parts, denoted by $8\pi GT_{ab}^{(M)}$ and $8\pi GT_{ab}^{(L)}$, respectively [6, 15]:

$$8\pi GT_{ab}^{(M)} = -\frac{1}{\lambda^2} \left(g_{acd} g_{bcd} - \frac{1}{4} \delta_{ab} g_{cde} g_{cde} \right), \quad (4.40)$$

$$8\pi GT_{ab}^{(L)} = \frac{1}{2\lambda^2} \left(\rho_a \rho_b - \Psi_a \Psi_b - \frac{1}{2} \delta_{ab} (\rho_c^2 - \Psi_c^2) \right), \quad (4.41)$$

$$\text{where} \quad \rho_a \equiv g_{bab}, \quad \Psi_a \equiv -\frac{1}{2} \varepsilon^{abcd} g_{bcd}. \quad (4.42)$$

The first energy-momentum tensor (4.40) is traceless, i.e. $8\pi GT_{aa}^{(M)} = 0$, which is a consequence of the identity $\eta_{ab}^i \bar{\eta}_{ab}^j = 0$ when applied to the first part of (4.38). The Ricci scalar $R \equiv R_{aa}$ can be calculated by (4.41) to yield

$$R = \frac{1}{2\lambda^2} (\rho_a^2 - \Psi_a^2). \quad (4.43)$$

The equation (4.43) immediately leads to the conclusion that a four-manifold emergent from pure symplectic gauge fields (without source terms) can have a vanishing Ricci scalar if and only if (see eqn. (4.35) and (4.36) and its derivation)

$$\rho_a = \pm \Psi_a \quad (4.44)$$

that is similar to the self-duality equation. When the relation (4.44) is obeyed, the second energy-momentum tensor $8\pi GT_{ab}^{(L)}$ (4.41) identically vanishes which confirms that the space of a Euclidean Schwarzschild blackhole is complete vacuum with no matter present.

5. Topological Invariants

In gravity topology can play a role at various levels. At the macroscopic level one may consider multiplying corrected universes and wormholes, whilst at the microscopic Planck scale spacetime topology may subject to quantum fluctuations; in analogy with others QFTs like sigma models and Yang-Mills theories, it is expected that the quantum tunneling process between different topologies are dominated by finite-action solutions of Euclidean gravity, the gravitational instantons.

One way to characterize topologically non-trivial solutions of the gravitational field equations is by the value of topologically invariant integral over certain polynomials of the curvature tensor. In four dimensions there are essentially two independent topological invariants the Euler Characteristics and the Hirzebruch signature.

Every manifold with an associated metric has topological invariants that characterize it. They hint at geometric similarities between manifolds sharing the same invariant. Here, we will calculate two such topological invariants of the Euclidean Schwarzschild metric.

5.1 Euler characteristic

We can use the Riemann tensor components to compute the Euler characteristic given by:

$$\chi(M) = \frac{1}{32\pi^2} \int_M \varepsilon^{abcd} R_{ab} \wedge R_{cd} + \frac{1}{16\pi^2} \int_{\partial M} \varepsilon^{abcd} \left(\theta_{ab} \wedge R_{cd} - \frac{2}{3} \theta_{ab} \wedge \theta_{cp} \wedge \theta_{pd} \right) \quad (5.1.1)$$

where θ_{AB} is the second fundamental form of the boundary ∂M . It is defined by

$$\theta_{AB} = \omega_{AB} - \omega_{0AB}, \quad (5.1.2)$$

where ω_{AB} are the actual connection 1-forms and ω_{0AB} are the connection 1-forms if the metric were locally a product form near the boundary [19]. The connection 1-form ω_{0AB} will have only tangential components on ∂M and so the second fundamental form θ_{AB} will have only normal components on ∂M .

The bulk part of the Euler characteristic is given by:

$$\chi_{bulk} = \frac{1}{32\pi^2} \int_M \varepsilon^{abcd} R_{ab} \wedge R_{cd} \quad (5.1.3)$$

To compute the expression in (5.1.3), we only need to consider 6 combinations, where one half is equivalent to the other half. These combinations are given as:

$$\begin{aligned} R_{12} \wedge R_{34} &= R_{34} \wedge R_{12} \\ R_{13} \wedge R_{24} &= R_{24} \wedge R_{13} \\ R_{14} \wedge R_{23} &= R_{23} \wedge R_{14} \end{aligned} \quad (5.1.4)$$

Since each permutation of 2 index pairs yields 2 combinations, and as shown in (5.1.4), equivalent pairs of combinations exist, we can say that (5.1.3) reduces to:

$$\chi_{bulk} = \frac{1}{4\pi^2} \int_M \left(\varepsilon^{1234} R_{12} \wedge R_{34} + \varepsilon^{1324} R_{13} \wedge R_{24} + \varepsilon^{1423} R_{14} \wedge R_{23} \right) \quad (5.1.5)$$

We can use the Bianchi identity for curvature tensor to show that:

$$dR^a{}_b = 0 \quad \Rightarrow \quad d(\omega^a{}_c \wedge \omega^c{}_b) = 0 \quad (5.1.6)$$

$$R_{ab} \wedge R_{cd} = d\omega_{ab} \wedge R_{cd} + \omega_{ap} \wedge \omega^p{}_b \wedge R_{cd}$$

$$d\omega_{ab} \wedge R_{cd} = d(\omega_{ab} \wedge R_{cd}) + \omega_{ab} \wedge dR_{cd} = d(\omega_{ab} \wedge R_{cd}) \quad (5.1.7)$$

$$\omega_{am} \wedge \omega^m{}_b \wedge R_{cd} = \omega_{ap} \wedge \omega^p{}_b \wedge d\omega_{cd} + \omega_{ap} \wedge \omega^p{}_b \wedge \omega_{cq} \wedge \omega^q{}_d \quad (5.1.8)$$

$$\begin{aligned} \omega_{am} \wedge \omega^m{}_b \wedge d\omega_{cd} &= d(\omega_{ap} \wedge \omega^p{}_b \wedge \omega_{cd}) + \underbrace{d(\omega_{ap} \wedge \omega^p{}_b)}_0 \wedge \omega_{cd} \\ \Rightarrow \quad \omega_{ap} \wedge \omega^p{}_b \wedge d\omega_{cd} &= d(\omega_{ap} \wedge \omega^p{}_b \wedge \omega_{cd}) \end{aligned} \quad (5.1.9)$$

$$\therefore \quad \omega_{ap} \wedge \omega^p_b \wedge R_{cd} = d(\omega_{ap} \wedge \omega^p_b \wedge \omega_{cd}) + \omega_{ap} \wedge \omega^p_b \wedge \omega_{cq} \wedge \omega^q_d \quad (5.1.10)$$

$$\begin{aligned} \int_M R_{ab} \wedge R_{cd} &= \int_M d(\omega_{ab} \wedge R_{cd} + \omega_{am} \wedge \omega^m_b \wedge \omega_{cd}) + \int_M \omega_{am} \wedge \omega^m_b \wedge \omega_{cn} \wedge \omega^n_d \\ &= \int_{\partial M} (\omega_{ab} \wedge R_{cd} + \omega_{am} \wedge \omega^m_b \wedge \omega_{cd}) + \int_M \omega_{am} \wedge \omega^m_b \wedge \omega_{cn} \wedge \omega^n_d \end{aligned} \quad (5.1.11)$$

We can see that for the 2nd term in (5.1.10) and for the 3rd term in (5.1.11) that:

$$\varepsilon^{abcd} \omega_{ap} \wedge \omega^p_b = \varepsilon^{abcd} (\omega_{ac} \wedge \omega^c_b + \omega_{ad} \wedge \omega^d_b) \quad (5.1.12)$$

$$\begin{aligned} \therefore \quad \varepsilon^{abcd} \omega_{ap} \wedge \omega^p_b \wedge \omega_{cq} \wedge \omega^q_d &= \varepsilon^{abcd} (\omega_{ac} \wedge \omega^c_b + \omega_{ad} \wedge \omega^d_b) \wedge (\omega_{ca} \wedge \omega^a_d + \omega_{cb} \wedge \omega^b_d) \\ &= 0 \end{aligned} \quad (5.1.13)$$

Using (5.1.13) we can see that (5.1.11) becomes:

$$\int_M R_{ab} \wedge R_{cd} = \int_{\partial M} (\omega_{ab} \wedge R_{cd} + \omega_{am} \wedge \omega^m_b \wedge \omega_{cd}) \quad (5.1.14)$$

For the 2nd term, we refer to (4.6) to point out that besides the 2nd row and column, all other rows and columns have only 2 non-zero elements (the first one has only one). ie.:

$$\begin{aligned} \sum_m \varepsilon^{abcd} \omega_{ap} \wedge \omega^p_b \wedge \omega_{cd} &= \varepsilon^{abcd} (\omega_{ac} \wedge \omega^c_b \wedge \omega_{cd} + \omega_{ad} \wedge \omega^d_b \wedge \omega_{cd}) \\ &= 0; \quad \forall \quad c, d \neq 2 \end{aligned} \quad (5.1.15)$$

Thus, the different non-vanishing components of (5.1.14) are:

$$\int_M R_{12} \wedge R_{34} = \int_{\partial M} \omega_{12} \wedge R_{34} = \int_{\partial M} \frac{2m^2}{r^3} dt \wedge d\theta \wedge \sin \theta d\phi \quad (5.1.16)$$

$$\begin{aligned} \int_M R_{13} \wedge R_{24} &= - \int_{\partial M} \omega_{12} \wedge \omega^2_3 \wedge \omega_{24} \\ &= - \int_{\partial M} \frac{m}{r^2} \left(1 - \frac{2m}{r}\right) dt \wedge d\theta \wedge \sin \theta d\phi \xrightarrow{r=2m} 0 \end{aligned} \quad (5.1.17)$$

$$\begin{aligned} \int_M R_{14} \wedge R_{23} &= \int_{\partial M} \omega_{12} \wedge \omega^2_4 \wedge \omega_{23} \\ &= - \int_{\partial M} \frac{m}{r^2} \left(1 - \frac{2m}{r}\right) dt \wedge d\theta \wedge \sin \theta d\phi \xrightarrow{r=2m} 0 \end{aligned} \quad (5.1.18)$$

Applying (5.1.14), (5.1.16), (5.1.17) and (5.1.18) to (5.1.5) gives us:

$$\begin{aligned} \chi_{bulk} &= \frac{1}{4\pi^2} \int_{\partial M} \omega_{12} \wedge R_{34} = \frac{1}{4\pi^2} \frac{2m^2}{r_h^3} \int_{\partial M} dt \wedge d\theta \wedge \sin \theta d\phi \\ &= \frac{1}{4\pi^2} \frac{2m^2}{r_h^3} \int_0^\beta dt \wedge \int_0^\pi \sin \theta d\theta \wedge \int_0^{2\pi} d\phi \\ \therefore \quad \chi_{bulk} &= \frac{2m^2}{\pi r_h^3} \beta \end{aligned} \quad (5.1.19)$$

Here, we compactify the imaginary time, such that it lies within the range: $0 \leq t \leq \beta$ (generalization of the condition of the removal of conical singularity for our class of metrics).

The upper limit β (realized as inverse temperature for the black hole) is given by:

$$\kappa\beta = 2\pi \quad \text{where} \quad \kappa = \frac{1}{2} \frac{\partial_r g_{tt}}{\sqrt{g_{tt}g_{rr}}} \Big|_{r=r_h} = \frac{1}{2} (\partial_r f(r))_{r=r_h} = \frac{m}{r_h^2} \quad (5.1.20)$$

$$\therefore \beta = \frac{2\pi}{\kappa} = \frac{2\pi r_h^2}{m} \quad (5.1.21)$$

Thus, for Schwarzschild, $r_h = 2m$ and applying (5.1.21) in (5.1.19) the bulk part of Euler characteristic is:

$$\chi_{bulk} = \frac{4m}{r_h} = 2 \quad (5.1.22)$$

The boundary integral term of the Euler characteristics is given by:

$$\chi_{boundary} = \frac{1}{16\pi^2} \int_{\partial M} \varepsilon^{abcd} \left(\theta_{ab} \wedge R_{cd} - \frac{2}{3} \theta_{ab} \wedge \theta_{cp} \wedge \theta_{pd} \right) \quad (5.1.23)$$

Recall that, the 1-form θ_{ab} is given by:

$$\theta_{ab} = \omega_{ab} - \omega_{0ab}, \quad \text{where} \quad \omega_{0ab} = \left(\omega_{ab} \right)_{r=\infty} \quad (5.1.24)$$

Only the component along the normal to the surface is to be treated differently ie.:

$$\theta_{12} = \omega_{12} \quad (5.1.25)$$

The θ^a_b matrix is given by:

$$\theta^a_b = \begin{pmatrix} 0 & \frac{m}{r^2} dt & 0 & 0 \\ -\frac{m}{r^2} dt & 0 & \left(1 - \sqrt{1 - \frac{2m}{r}}\right) d\theta & \left(1 - \sqrt{1 - \frac{2m}{r}}\right) \sin \theta d\varphi \\ 0 & -\left(1 - \sqrt{1 - \frac{2m}{r}}\right) d\theta & 0 & 0 \\ 0 & -\left(1 - \sqrt{1 - \frac{2m}{r}}\right) \sin \theta d\varphi & 0 & 0 \end{pmatrix} \quad (5.1.26)$$

In this case, since $\partial M \Rightarrow r = \infty$, when θ_{12} vanishes as $r \rightarrow \infty$. Thus, we can effectively say, $\theta_{ab} = 0$ which corresponds to setting $\chi_{boundary} = 0$ so that we can write:

$$\chi(M) = \chi_{bulk} + \chi_{boundary} = 2 + 0 = 2 \quad (5.1.27)$$

which is the value of Euler characteristic for Euclidean Schwarzschild metric. (see also [20] for a similar computation which was reported there for the first time.)

Recalling how the Schwarzschild metric is a sum of an $SU(2)_L$ instanton and $SU(2)_R$ anti-instanton resulting from the $SU(2)_+$ and $SU(2)_-$ gauge fields (described in the appendix (7.3.4), (7.3.5) and (7.3.6)), we can further calculate the Euler characteristics using:

$$\eta_{\mu\nu}^{(\pm)i} \eta_{\lambda\gamma}^{(\pm)i} = \delta_{\mu\lambda} \delta_{\nu\gamma} - \delta_{\mu\gamma} \delta_{\nu\lambda} \pm \varepsilon_{\mu\nu\lambda\gamma} \quad (5.1.28)$$

$$\varepsilon_{\mu\nu\lambda\gamma} = \frac{1}{2} \left(\eta_{\mu\nu}^{(+i)} \eta_{\lambda\gamma}^{(+i)} - \eta_{\mu\nu}^{(-i)} \eta_{\lambda\gamma}^{(-i)} \right) \quad (5.1.29)$$

Thus (5.1.3) reduces to

$$\begin{aligned} \frac{1}{32\pi^2} \int_M \varepsilon^{abcd} R_{ab} \wedge R_{cd} &= \frac{1}{64\pi^2} \int_M \left(\eta_{\mu\nu}^{(+i)} R_{ab} \wedge \eta_{\lambda\gamma}^{(+i)} R_{cd} - \eta_{\mu\nu}^{(-i)} R_{ab} \wedge \eta_{\lambda\gamma}^{(-i)} R_{cd} \right) \\ &= \frac{1}{4\pi^2} \int_M \left(F^{(+i)} \wedge F^{(+i)} - F^{(-i)} \wedge F^{(-i)} \right) \end{aligned} \quad (5.1.30)$$

It is straightforward to express the topological invariant in terms of $SU(2)$ gauge fields.

$$\therefore \quad \chi_{bulk} = \frac{1}{4\pi^2} \int_M \left(F^{(+i)} \wedge F^{(+i)} - F^{(-i)} \wedge F^{(-i)} \right) \quad (5.1.31)$$

We could now follow the same process as before invoking Stoke's theorem and convert (5.1.31) into a boundary integral using (5.1.14) to obtain:

$$\begin{aligned} \frac{\varepsilon^{abcd}}{32\pi^2} \int_M R_{ab} \wedge R_{cd} &= \frac{\varepsilon^{abcd}}{32\pi^2} \int_{\partial M} (\omega_{ab} \wedge R_{cd} + \omega_{am} \wedge \omega_b^m \wedge \omega_{cd}) \\ &= \frac{1}{4\pi^2} \int_{\partial M} \left(A^{(+i)} \wedge F^{(+i)} - A^{(-i)} \wedge F^{(-i)} \right) + \frac{\varepsilon^{abcd}}{32\pi^2} \int_{\partial M} \omega_{am} \wedge \omega_b^m \wedge \omega_{cd} \end{aligned}$$

Seeing how the 2nd integrand vanishes for most combinations, and otherwise vanishes on the boundary itself, we can focus on the 1st integrand alone.

$$\chi_{bulk} = \frac{1}{4\pi^2} \int_{\partial M} \left(A^{(+i)} \wedge F^{(+i)} - A^{(-i)} \wedge F^{(-i)} \right) = \chi_{bulk}^+ + \chi_{bulk}^- \quad (5.1.32)$$

Thus, we can compute the Euler character bulk values using (4.24) and (4.25) as:

$$\begin{aligned} \chi_{bulk}^+ &= \frac{1}{4\pi^2} \int_{\partial M} A^{(+i)} \wedge F^{(+i)} \\ &= \frac{1}{4\pi^2} \int_{\partial M} \frac{m}{4r^4} (be^3 \wedge z - be^4 \wedge y + 2ae^1 \wedge \bar{x} - 2ce^4 \wedge x) \\ &= \frac{1}{4\pi^2} \int_{\partial M} \frac{m}{4r^4} \left\{ (2a - 2b) e^1 \wedge e^3 \wedge e^4 - 2c e^1 \wedge e^2 \wedge e^4 \right\} \\ &= \frac{1}{4\pi^2} \left\{ \frac{m^2}{2r_h^3} \int_{\partial M} dt \wedge \sin \theta d\theta \wedge d\varphi - \frac{m}{2} \cos \theta|_0^\pi \int_{\partial M} \frac{1}{r^3} dt \wedge dr \wedge d\varphi \right\} \\ &= \frac{m^2}{2r_h^3 \pi} \beta + \frac{m}{4r_h^2 \pi} \beta = \left(\frac{1}{16m\pi} + \frac{1}{16m\pi} \right) \beta = 1 \end{aligned} \quad (5.1.33)$$

$$\begin{aligned} \chi_{bulk}^- &= -\frac{1}{4\pi^2} \int_{\partial M} A^{(-i)} \wedge F^{(-i)} \\ &= -\frac{1}{4\pi^2} \int_{\partial M} -\frac{m}{4r^4} (be^3 \wedge z - be^4 \wedge y + 2ae^1 \wedge \bar{x} - 2ce^4 \wedge x) = 1 \end{aligned} \quad (5.1.34)$$

For verification, we evaluate the contributions according to (5.1.31) using (4.25) to get:

$$\begin{aligned}
\chi_{bulk}^+ &= \frac{1}{4\pi^2} \int_M F^{(+i)} \wedge F^{(+i)} \\
&= \frac{1}{4\pi^2} \int_M \left(\frac{m^2}{2r^6} + \frac{m^2}{2r^6} + \frac{2m^2}{r^6} \right) \nu = \frac{3m^2}{4\pi^2} \int_M \frac{1}{r^6} \nu \\
&= \frac{m^2}{4\pi^2} \int_M d \left(\frac{1}{r^3} dt \wedge d\theta \wedge \sin \theta d\varphi \right) = \frac{m^2}{4\pi^2} \int_{\partial M} \frac{1}{r^3} dt \wedge d\theta \wedge \sin \theta d\varphi \\
&= \frac{m^2}{4\pi^2 r_h^3} \int_{\partial M} dt \wedge d\theta \wedge \sin \theta d\varphi = \frac{m^2}{r_h^3 \pi} \beta = \frac{2m}{r_h} = 1
\end{aligned} \tag{5.1.35}$$

$$\begin{aligned}
\chi_{bulk}^- &= -\frac{1}{4\pi^2} \int_M F^{(-i)} \wedge F^{(-i)} \\
&= -\frac{1}{4\pi^2} \int_M -\left(\frac{m^2}{2r^6} + \frac{m^2}{2r^6} + \frac{2m^2}{r^6} \right) \nu = 1
\end{aligned} \tag{5.1.36}$$

Thus, we can clearly see that the overall bulk value of the Euler characteristic is the sum of the two individual values due to $SU(2)_+$ and $SU(2)_-$ gauge fields, giving:

$$\chi_{bulk} = \chi_{bulk}^+ + \chi_{bulk}^- = 1 + 1 = 2 \tag{5.1.37}$$

This also shows both gauge fields contributing eqally to the overall Euler invariant.

5.2 Hirzebruch signature complex

Now we turn our attention to the other topological invariant, the **Hirzebruch signature**. It's formula is given by:

$$\tau(M) = -\frac{1}{24\pi^2} \left(\int_M \text{Tr } R \wedge R + \int_{\partial M} \text{Tr } \theta \wedge R + \eta_S(\partial M) \right) \tag{5.2.1}$$

The bulk part of the integral (5.2.1) can be given as:

$$\tau_{bulk} = -\frac{1}{24\pi^2} \int_M \text{Tr } R \wedge R = -\frac{1}{24\pi^2} \int_M R_{ab} \wedge R^{ab} \tag{5.2.2}$$

However, we can see from (4.15) that every element of the curvature 2-forms has a single 2-form term. Thus, we can write:

$$R_{ab} \wedge R^{ab} = 0 \quad \Rightarrow \quad \tau_{bulk} = 0 \tag{5.2.3}$$

Now, as seen in the case of $\chi(M)$, the boundary integral term here will also vanish following the same logic.

$$\theta_{ab} \wedge R^{ab} = 0 \quad \Rightarrow \quad \tau_{boundary} = 0 \tag{5.2.4}$$

This leaves us with nothing but the last term, known as the spectral asymmetry term $\eta_S(\partial M)$ which in this case is also known to vanish. Therefore:

$$\tau(M) = 0 \tag{5.2.5}$$

As before, analyzing from the point of view of $SU(2)_\pm$ gauge fields lets us use:

$$\delta_{\mu\lambda}\delta_{\nu\gamma} - \delta_{\mu\gamma}\delta_{\nu\lambda} = \frac{1}{2} \left(\eta_{\mu\nu}^{(+i)} \eta_{\lambda\gamma}^{(+i)} + \eta_{\mu\nu}^{(-i)} \eta_{\lambda\gamma}^{(-i)} \right) \quad (5.2.6)$$

to write the bulk part of the signature complex as

$$\begin{aligned} \tau_{bulk} &= -\frac{1}{24\pi^2} \int_M \text{Tr}(R \wedge R) = -\frac{1}{24\pi^2} \int_M \delta^{ac} \delta^{bd} R_{ab} \wedge R_{cd} \\ &= -\frac{1}{48\pi^2} \int_M \left(\delta^{ac} \delta^{bd} R_{ab} \wedge R_{cd} + \delta^{ad} \delta^{bc} R_{ab} \wedge R_{cd} \right) \\ &= -\frac{1}{48\pi^2} \int_M \left(\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc} \right) R_{ab} \wedge R_{cd} \\ &= -\frac{1}{96\pi^2} \int_M \left(\eta_{ab}^{(+i)} \eta_{cd}^{(+i)} + \eta_{ab}^{(-i)} \eta_{cd}^{(-i)} \right) R_{ab} \wedge R_{cd} \\ &= -\frac{1}{6\pi^2} \int_M \left(F^{(+i)} \wedge F^{(+i)} + F^{(-i)} \wedge F^{(-i)} \right) \\ &= -\frac{2}{3} \left(\chi_{bulk}^+ - \chi_{bulk}^- \right) = \frac{2}{3} (-1 + 1) = 0 \end{aligned} \quad (5.2.7)$$

where we can see that the individual bulk contribution is:

$$\tau_{bulk} = \tau_{bulk}^+ + \tau_{bulk}^- \quad (5.2.8)$$

$$\tau_{bulk}^+ = -\frac{2}{3} \chi_{bulk}^+ = -\frac{2}{3} \quad \tau_{bulk}^- = \frac{2}{3} \chi_{bulk}^- = \frac{2}{3} \quad (5.2.9)$$

6. Discussion

In this note we have started applying the bottom-up approach of emergent gravity to (Euclidean) Schwarzschild black hole which we dub as emergent Schwarzschild. The emergent Schwarzschild solution describes a Ricci-flat manifold, although it is not a Kähler manifold. So it does not admit a natural symplectic structure. The best alternative choice as was utilized by Etesi and Hausel [21]) was to consider the (anti) self-dual harmonic two-forms on the space and define a Poisson algebra determined by the self-dual harmonic two-form. However a magnetic mass (and an electric mass) at the origin seems to violate the Jacobi identity of the underlying Poisson algebra. Therefore the Schwarzschild black hole remained a challenging goal to pursue from the bottom-up approaches of emergent gravity.

What we have been able to do is to find a suitable Darboux chart for the Schwarzschild solution for which locally we have the Jacobi identity satisfied for the symplectic $U(1)$ gauge fields emergent from the metric as well as the Bianchi identity for the vector fields. We set up the Seiberg Witten map between the commutative and non-commutative description and did a thorough geometrical engineering for the emergent Schwarzschild solution. We saw that the two instantons forming the emergent Schwarzschild solution belong to different gauge groups namely $SU(2)_L$ and $SU(2)_R$ and hence they can't decay into a vacuum thus explaining the stability of emergent Schwarzschild space against perturbation, which might

be a generic fact for any Ricci-flat four manifold as ours. Hence we can definitely conclude that the Euclidean Schwarzschild metric is **not** a gravitational instanton.

It will be interesting to investigate how to analyze a charged black hole solution in this bottom-up approach of emergent gravity. In [22], it was suggested that there are two kinds of 4D and 2D EBHs in the nature. The first kind of extremal black holes can be obtained by first taking the extreme limit and then the boundary limit starting from general non-extremal configuration. The entropy of this kind of EBH is zero. The second kind of EBH which still holds the topological configuration of NEBH can be obtained by taking the boundary limit first and then the extreme limit. The entropy of this kind of EBH satisfies the BH formula. These two kinds of EBHs have different intrinsic thermodynamical properties owing to different topological characteristics playing an essential role in the classification of these solutions. For the first kind, the Euler characteristic is zero; and for the second, it is equal to two or one provided they are 4D or 2D EBHs respectively. Now it will be interesting to address the fact whether such a change in topology of spacetime can be explained from the point of view of a well-defined mechanism inspired by the emergent gravity approach which was set up by one of the authors in [23]. An article covering the emergent nature and further aspect of Taub-NUT and its connection with dynamical systems is going to appear soon ([24])

Acknowledgments

The research of RR was supported by FAPESP through Instituto de Física, Universidade de Sao Paulo with grant number 2013/17765-0. This work was performed during RR's visit to S.N. Bose National Centre for Basic Sciences in Kolkata. He thanks SNBNCBS for the hospitality and support during that period.

7. APPENDIX

7.1 Relations from Differential Geometry

The Cartan structure equations are powerful tools in differential geometry, useful for the analysis of curvature involved in General Relativity. Cartan's first structure equation is:

$$T^a = de^a + \omega^a_b \wedge e^b \quad (7.1.1)$$

Under torsion free condition ($T^a = 0$), we have:

$$de^a = -\omega^a_b \wedge e^b \quad \Rightarrow \quad \partial_\mu e^a_\nu = -\omega_\mu^a_b e^b_\nu \quad (7.1.2)$$

Upon contraction with E_c^ν , we can proceed to write:

$$\begin{aligned} E_c^\nu \partial_\mu e^a_\nu &= -\omega_\mu^a_b (e^b_\nu E_c^\nu) \\ \Rightarrow \quad \partial_\mu (E_c^\nu e^a_\nu) - e^a_\nu \partial_\mu E_c^\nu &= -\omega_\mu^a_b \delta_c^b \\ \Rightarrow \quad \underbrace{\partial_\mu (\delta_c^a)}_0 - e^a_\nu \partial_\mu E_c^\nu &= -\omega_\mu^a_c \\ \Rightarrow \quad e^a_\nu \partial_\mu E_c^\nu &= \omega_\mu^a_c \\ \therefore \quad \partial_\mu E_c^\nu &= \omega_\mu^a_c E_a^\nu \end{aligned} \quad (7.1.3)$$

7.2 Permutation operations with the t'Hooft symbols

Since each of the t'Hooft symbols has only one non-zero element in each row and column, they are permutation matrices. Here, we will just establish the permutation rules associated with each of the matrices. The t'Hooft symbols are given by:

$$\eta^{(+1)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \eta^{(+2)} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \eta^{(+3)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (7.2.1)$$

$$\eta^{(-1)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \eta^{(-2)} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \eta^{(-3)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (7.2.2)$$

Suppose the arrangement with the original matrix in terms of rows is 1234. Then, left-multiplication with each of the t'Hooft symbols gives the following:

t'Hooft Permutations			
Symbol	η^1	η^2	η^3
$SU(2)_+$	43 $\bar{2}$ $\bar{1}$	34 $\bar{1}$ $\bar{2}$	2 $\bar{1}$ 4 $\bar{3}$
$SU(2)_-$	$\bar{4}$ 3 $\bar{2}$ 1	3 $\bar{4}$ $\bar{1}$ 2	2 $\bar{1}$ $\bar{4}$ 3

where numbers labelled as \bar{X} are rows where the sign has been flipped.

7.3 $SU(2)_\pm$ gauge fields

The various gauge fields involved with equation (4.23) are given by:

$$\begin{aligned}
A^{(+1)} &= -\frac{1}{4r} \text{Tr} \begin{pmatrix} 0 & b e^4 & c e^4 & 0 \\ 0 & b e^3 & 0 & -c e^4 \\ a e^1 & 0 & b e^3 & b e^4 \\ 0 & -a e^1 & 0 & 0 \end{pmatrix} & A^{(-1)} &= -\frac{1}{4r} \text{Tr} \begin{pmatrix} 0 & -b e^4 & -c e^4 & 0 \\ 0 & b e^3 & 0 & -c e^4 \\ a e^1 & 0 & b e^3 & b e^4 \\ 0 & a e^1 & 0 & 0 \end{pmatrix} \\
&= -\frac{1}{2r} b e^3 & & = -\frac{1}{2r} b e^3 \quad (7.3.1)
\end{aligned}$$

$$\begin{aligned}
A^{(+2)} &= -\frac{1}{4r} \text{Tr} \begin{pmatrix} 0 & -b e^3 & 0 & c e^4 \\ 0 & b e^4 & c e^4 & 0 \\ 0 & a e^1 & 0 & 0 \\ a e^1 & 0 & b e^3 & b e^4 \end{pmatrix} & A^{(-2)} &= -\frac{1}{4r} \text{Tr} \begin{pmatrix} 0 & -b e^3 & 0 & c e^4 \\ 0 & -b e^4 & -c e^4 & 0 \\ 0 & a e^1 & 0 & 0 \\ -a e^1 & 0 & -b e^3 & -b e^4 \end{pmatrix} \\
&= -\frac{1}{2r} b e^4 & & = -\frac{1}{2r} b e^4 \quad (7.3.2)
\end{aligned}$$

$$\begin{aligned}
A^{(+3)} &= -\frac{1}{4r} \text{Tr} \begin{pmatrix} -a e^1 & 0 & -b e^3 & -b e^4 \\ 0 & -a e^1 & 0 & 0 \\ 0 & b e^4 & c e^4 & 0 \\ 0 & -b e^3 & 0 & c e^4 \end{pmatrix} & A^{(-3)} &= -\frac{1}{4r} \text{Tr} \begin{pmatrix} -a e^1 & 0 & -b e^3 & -b e^4 \\ 0 & -a e^1 & 0 & 0 \\ 0 & -b e^4 & -c e^4 & 0 \\ 0 & b e^3 & 0 & -c e^4 \end{pmatrix} \\
&= \frac{1}{2r} (a e^1 - c e^4) & & = \frac{1}{2r} (a e^1 + c e^4) \quad (7.3.3)
\end{aligned}$$

$$\begin{aligned}
F^{(+1)} &= -\frac{m}{4r^3} \text{Tr} \begin{pmatrix} z & -\bar{y} & -2\bar{x} & 0 \\ y & \bar{z} & 0 & 2\bar{x} \\ 2x & 0 & \bar{z} & -\bar{y} \\ 0 & -2x & y & z \end{pmatrix} & F^{(-1)} &= -\frac{m}{4r^3} \text{Tr} \begin{pmatrix} -z & \bar{y} & 2\bar{x} & 0 \\ y & \bar{z} & 0 & 2\bar{x} \\ 2x & 0 & \bar{z} & \bar{y} \\ 0 & 2x & -y & -z \end{pmatrix} \\
&= -\frac{m}{2r^3} (z + \bar{z}) & & = \frac{m}{2r^3} (z - \bar{z}) \quad (7.3.4)
\end{aligned}$$

$$\begin{aligned}
F^{(+2)} &= -\frac{m}{4r^3} \text{Tr} \begin{pmatrix} -y & -\bar{z} & 0 & -2\bar{x} \\ z & -\bar{y} & -2\bar{x} & 0 \\ 0 & 2x & -y & -z \\ 2x & 0 & \bar{z} & -\bar{y} \end{pmatrix} & F^{(-2)} &= -\frac{m}{4r^3} \text{Tr} \begin{pmatrix} -y & -\bar{z} & 0 & -2\bar{x} \\ -z & \bar{y} & 2\bar{x} & 0 \\ 0 & 2x & -y & -z \\ -2x & 0 & -\bar{z} & \bar{y} \end{pmatrix} \\
&= \frac{m}{2r^3} (y + \bar{y}) & & = \frac{m}{2r^3} (y - \bar{y}) \quad (7.3.5)
\end{aligned}$$

$$\begin{aligned}
F^{(+3)} &= -\frac{m}{4r^3} \text{Tr} \begin{pmatrix} -2x & 0 & -\bar{z} & \bar{y} \\ 0 & -2x & y & z \\ z & -\bar{y} & -2\bar{x} & 0 \\ -y & -\bar{z} & 0 & -2\bar{x} \end{pmatrix} & F^{(-3)} &= -\frac{m}{4r^3} \text{Tr} \begin{pmatrix} -2x & 0 & -\bar{z} & -\bar{y} \\ 0 & -2x & y & z \\ -z & \bar{y} & 2\bar{x} & 0 \\ y & \bar{z} & 0 & 2\bar{x} \end{pmatrix} \\
&= \frac{m}{r^3} (x + \bar{x}) & & = \frac{m}{r^3} (x - \bar{x}) \quad (7.3.6)
\end{aligned}$$

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